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# A Graph-theoretic Pipe Network Method for water flow simulation in a porous medium: GPNM

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ABSTRACT

A new numerical simulation method for water flow in a porous medium is proposed. A porous medium is discretized graph-theoretically into a discrete pipe network. Each pipe in the oriented network is defined as a weighted element with a starting node and an ending node. Equivalent hydraulic parameters are derived based on the Darcy's Law. A node law of flow rate and a pipe law of pressure are derived based on the conservation of mass and energy, as well as the graph-theoretic network theory. A unified governing equation for both the inner pipes and the boundary pipes are deduced. A conversion law of flow rate/velocity is proposed and discussed. A few case studies are analyzed and compared with those from analytical solutions and finite element analysis. It shows that the proposed Graph-theoretic Pipe Network Method (GPNM) is effective in analyzing water flow in a porous medium. The advantage of the proposed GPNM is that a continuous porous medium is discretized into a discrete pipe network, which is analyzed same as for a discrete fracture network. Solutions of water pressures and flow rates in the discrete pipe network are obtained by solving a system of nonhomogeneous linear equations. It is demonstrated with high efficiency and accuracy. The developed method can be extended to analyzing water flow in fractured and porous media in 3-D conditions.



## 1. Introduction

Water flow in a porous medium is a fundamental and important issue in geotechnical and hydrogeological engineering, underground tunneling and mining, dam engineering, nuclear waste disposal, oil and gas reposition, etc. A porous medium consist of solid matrix and pores network, which is one of the most prevalent structures of materials in nature. Continuous models are often established by simplification of the matrix and three kinds of heterogeneities, i.e., the microscopic (pore scale), the macroscopic (core plug scale) and the megascopic (field scale) pores (Abou Najm et al., 2010; Arora et al., 2011; Sahimi, 2011). Equivalent continuous models can also be derived for a fractured medium based on the concepts of representative elementary volume (REV). The simplification is assumed not to change the macroscopic properties of a porous medium. The equivalent continuous medium is then discretized into finite elements for conducting water flow simulations.

Conventional simulation methods for water flow in a porous medium are mainly based on the finite difference method (FDM), the finite element method (FEM) and the finite volume method (FVM), such as some widely used commercial softwares of MODFLOW (based on FDM) (Kim et al., 2008; Langevin and Guo, 2006), COMSOL Multiphysics (based on FEM) (Xu et al., 2011a,b) and FLUENT (based on FVM) (Crandall et al., 2010; Xu and Jiang, 2008).

The FEM solves partial differential equations based on the variation principle and the weighted residual method, which considers the governing equations of nodes as well as the neighborhoods (Guo et al., 2009; Huyakorn et al., 1983; Zienkiewicz et al., 2005). The FDM only considers the nodes' values, but not the variation functions among different mesh nodes (Cooley, 1971; Jing, 2003; Narasimhan and Witherspoon, 1976). Similar to the FDM, the FVM only solves the node's values. Neighborhood equations are assumed to interpret variations among different mesh nodes, which are similar to the FEM (Demirdžić and Martinović, 1993; Lunati and Jenny, 2006, 2008). The FVM is often regarded as a bridge between the FDM and the FEM (Fallah et al., 2000; Jing and Hudson, 2002; Selmin, 1993). Generally, the FDM is easy to be implement, but the quality of the approximation between the grid nodes is poor, hence it suffers from a major drawback, not as flexible as the FEM in dealing with complex boundary conditions and material inhomogeneity (Jing, 2003). However, the FEM suffers

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**Nomenclature**

| | |
|---|---|
| $\mathbf{A}$ | incidence matrix |
| $\mathbf{A}^T$ | transpose of the incidence matrix |
| $\mathbf{A}_\Gamma$ | augmented incidence matrix |
| $a_{\Gamma ij}$ | element of the augmented incidence matrix |
| b$i$ | serial number of a pipe |
| $PN$ | total number of pipes |
| $C$ | constant |
| $C_{ijk}$ | circumcenter of the triangular mesh ($i$)–($j$)–($k$) |
| $F$ | area |
| $F_{ijk}$ | area of the triangular mesh ($i$)–($j$)–($k$) |
| $G$ | conductance |
| $g$ | gravity acceleration |
| $G_{bi}$ | conductance of pipe b$i$ |
| $G_{ij}$ | conductance of pipe ($i$)–($j$) |
| $Grad(P_j)_x$ | *X-component* of the total hydraulic pressure gradient at node ($j$) |
| $Grad(P_j)_y$ | *Y-component* of the total hydraulic pressure gradient at node ($j$) |
| $h_{jk}$ | corresponding altitude to pipe ($j$)–($k$) in the triangular mesh ($i$)–($j$)–($k$) |
| $h_1$ | elevation head |
| $h_2$ | pressure head |
| $h_3$ | velocity head |
| $I, J, K$ | degree of angles in triangular mesh ($i$)–($j$)–($k$) |
| ($i$) | serial number of a node |
| $k$ | permeability |
| $k_0$ | permeability of the medium without grouting |
| $k_g$ | permeability of the grouted medium |
| $KI, IJ, KJ$ | side length of the triangular mesh ($i$)–($j$)–($k$) |
| $L$ | length |
| $L^{ij}$ | length of the perpendicular bisector segment corresponding to pipe ($i$)–($j$) |
| $L_v^{ij}$ | upright projection of the perpendicular bisector segment to the direction of velocity at node ($j$) |
| $L_{jk}$ | length of pipe ($j$)–($k$) |
| $M_{ij}$ | center of pipe ($i$)–($j$) |
| $NN$ | total number of nodes |
| $P$ | total hydraulic pressure |
| $P_0$ | total hydraulic pressure on the tunnel surface |
| $P_1$ | total hydraulic pressure on the outer boundary of the grouting rim |
| $P_2$ | total hydraulic pressure on the model surface |
| $\mathbf{P}_b$ | total hydraulic pressure reduction matrix |
| $P_{bi}$ | total hydraulic pressure reduction in pipe b$i$ |
| $P_j$ | total hydraulic pressure at node ($j$) |
| $\mathbf{P}_n$ | total hydraulic pressure matrix |
| $P_r$ | total hydraulic pressure on the circle of radius $r$ |
| $Q_0$ | water flow through the tunnel face |
| $\mathbf{Q}_b$ | water flow matrix |
| $Q_{bi}$ | water flow in pipe b$i$ |
| $Q_F$ | water flow through area of $F$ |
| $Q_{i-jk}$ | water flow perpendicularly from node ($i$) to pipe ($j$)–($k$) |
| $Q_{jk}$ | water flow from node ($j$) to ($k$) |
| $Q_r$ | water flow through the circle *of radius r* |
| $r$ | radius |
| $R$ | resistance |
| $r_0$ | radius of the tunnel |
| $r_1$ | radius of the grouting rim |
| $r_2$ | radius of the model |
| $R_{ij}$ | resistance of pipe ($i$)–($j$) |
| $\mathbf{S}_b$ | mean source matrix |
| $U_j$ | velocity at node ($j$) |
| $U_{jx}$ | $X$-component of the velocity at node ($j$) |
| $U_{jy}$ | $Y$-component of the velocity at node ($j$) |
| $U_r$ | velocity through the circle of radius $r$ |
| $W$ | width |
| $x_i$ | $X$-coordinate of node ($i$) |
| $\mathbf{Y}_b$ | mean conductance matrix |
| $Y_{bij}$ | element of the mean conductance matrix |
| $y_i$ | $Y$-coordinate of node ($i$) |
| $\mathbf{Z}_b$ | mean resistance matrix |
| $Z_{bij}$ | element of the mean resistance matrix |
| $\alpha_{ji}$ | angle from the horizontal axis to pipe ($j$)–($i$) |
| $\beta_j$ | angle from the horizontal axis to the direction of velocity at node ($j$) |
| $\mu$ | water viscosity |
| $\mu_0$ | water viscosity in the medium without grouting |
| $\mu_g$ | water viscosity in the grouted medium |
| $\rho$ | water density |
| $\Delta P$ | total hydraulic pressure reduction |
| $\Delta P_{i-jk}$ | total hydraulic pressure reduction between node ($i$) and pipe ($j$)–($k$) when water runs perpendicularly from node ($i$) to pipe ($j$)–($k$) |

from another shortcoming of the excessive computational burden, hence the FDM and FVM tend to be used more often in numerical simulations of computational fluid dynamics (CFD) while very fine discretization and large numbers of mesh nodes are required in some CFD problems (Mishev, 1998; Sharma et al., 2012; Taylor et al., 1995). Especially, the FVM overcomes some shortcomings of the FDM, and combines some advantages of the FEM, as well as reduces the computational effort.

Another important and alternative CFD method is the lattice Boltzmann method (LBM). It originated from the lattice gas automata in 1980s (Eggels, 1996; Higuera and Jiménez, 1989; McNamara and Zanetti, 1988). It has been extraordinarily successful in many applications, including turbulent, multi-component, multi-phase fluid flows, solute transport, interfacial dynamics as well as complex boundaries, to name but a few (Aharonov and Rothman, 1993; Chau et al., 2005; Sukop and Or, 2004; Walsh and Saar, 2010). Instead of solving the Navier–Stokes (NS) equations and discretizing the macroscopic continuum equations, the lattice Boltzmann method solves the Boltzmann equation by discretizing the physical and velocity space with lattice nodes and a set of microscopic velocity vectors. Based on simulating streaming and collision processes across a limited number of particles, the averaged macroscopic behavior of viscous water flow is evinced by the intrinsic particle interactions (Attar and Körner, 2011; Frisch et al., 1986; Nourgaliev et al., 2003; Rothman and Zaleski, 1997). Different from other numerical methods, the LBM has three distinguishing features, which are discussed in reviews (Chen and Doolen, 1998). Firstly, the convection operator in velocity space is linear, and combined with a collision operator allows the recovery of the nonlinear macroscopic advection. Secondly, the incompressible NS equations can be obtained in the nearly incompressible limit of the LBM, and the pressure is calculated using an equation of state. Thirdly, a minimal set of velocities is utilized which greatly simplify the transformation relating the microscopic distribution function and macroscopic quantities. In a word, the LBM is based on kinetic theory and statistical mechanics, and represents the macroscopic responses of water flow through statistical and average properties of microscopic particles. By developing a