International Journal of Heat and Fluid Flow 32 (2011) 1120–1128

Contents lists available at SciVerse ScienceDirect

International Journal of Heat and Fluid Flow

journal homepage: www.elsevier.com/locate/ijhff

Numerical simulation of magnetohydrodynamic flow in a toroidal duct of square cross-section

S. Vantieghem *, B. Knaepen

Statistical and Plasma Physics, Université Libre de Bruxelles, Campus Plaine, CP 231, B-1050 Brussels, Belgium

ARTICLE INFO

Article history:
Received 13 August 2010
Received in revised form 13 June 2011
Accepted 31 August 2011
Available online 23 September 2011

Keywords:
Magnetohydrodynamics
Hartmann layer
Curved duct

ABSTRACT

We present numerical simulation results of the quasi-static magnetohydrodynamic (MHD) flow in a toroidal duct of square cross-section with insulating Hartmann walls and conducting side walls. Both laminar and turbulent flows are considered. In the case of steady flows, we present a comprehensive analysis of the secondary flow. It consists of two counter-rotating vortex cells, with additional side wall vortices emerging at sufficiently high Hartmann number. Our results agree well with existing asymptotic analysis. In the turbulent regime, we make a comparison between hydrodynamic and MHD flows. We find that the curvature induces an asymmetry between the inner and outer side of the duct, with higher turbulence intensities occurring at the outer side wall. The magnetic field is seen to stabilize the flow so that only the outer side layer remains unstable. These features are illustrated both by a study of statistically averaged quantities and by a visualization of (instantaneous) coherent vortices.

1. Introduction

In magnetohydrodynamics (MHD), one studies the coupling between flows of electrically conducting fluids and electromagnetic fields. This branch of physics describes a vast range of phenomena, like the origin of the Earth's magnetic field or the suppression of turbulence due to a magnetic field in industrial melt flows. For most industrial applications and laboratory flows, the coupling is virtually one-way; this means that the flow is significantly affected by the Lorentz force due to the action of the magnetic field, but that the induced magnetic field remains negligible compared to the externally imposed one. Such a behavior is the signature of flows in which the magnetic Reynolds number $R_m = \mu\sigma U L$ is small compared to one. Here, μ and σ are respectively the magnetic permeability and the electrical conductivity of the fluid, while U and L are typical velocity and length scales of the flow under consideration.

Under such conditions, called the quasi-static regime, the magnetic field mostly tends to suppress variations along its direction. If the magnetic field intensity is high, this results in a flow consisting of an extended, quasi-uniform core, surrounded by thin shear layers due to the presence of solid boundaries or discontinuities. We can distinguish between two different types of wall shear layers: the *Hartmann layer*, which occurs at walls with their normal vector non-perpendicular to the magnetic field and the *side layer* (or *parallel layer*), which emerges at walls parallel to the magnetic field. Under laminar conditions, their thickness can be expressed in terms of the Hartmann number M, a dimensionless measure of the ratio between the Lorentz and viscous force. The Hartmann layer has a typical thickness of $\mathcal{O}(M^{-1})$, while that of the side layer scales as $\mathcal{O}(M^{-1/2})$. These shear layers are prone to three-dimensional stabilities, as discussed by Thess and Zikanov (2007).

In the past years, there has been considerable interest in the role of the different shear layers in the transition of wall-bounded MHD shear flows. Krasnov et al. (2004) performed a computational study of the instability of the Hartmann layer, and found that the parameter which governs the transition, is the ratio between the Reynolds number Re and the Hartmann number. In their study, the transition is seen to occur for values of Re/M between 350 and 400. Moresco and Alboussière (2004) performed friction factor measurements in a toroidal duct of square cross-section at high Hartmann and Reynolds number. Since the major part of the friction occurs in the Hartmann layer for high Hartmann number flows, they conjectured that a sudden change in the behavior of the friction factor is related to a transition in the Hartmann layer. Their measurements showed that this transition occurs at $Re/M \approx 380$, regardless of the exact value of the Hartmann number. A linear stability analysis of Lingwood and Alboussière (1999) yielded a critical value of $Re/M \approx 48{,}250$. This large discrepancy indicates that the transition is triggered by non-linear effects.

The experimental method adopted in Moresco and Alboussière (2004) did however not allow to study the behavior of the side layers. A computational study of Krasnov et al. (2010) on the other hand, showed that the nuclei of instability in MHD straight duct flow are located in the side layers. The common feature in all these

* Corresponding author. Tel.: +32 2650 58 23.
E-mail address: stvtiegh@ulb.ac.be (S. Vantieghem).

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doi:10.1016/j.ijheatfluidflow.2011.08.004

studies is that the Hartmann walls are insulating. Other authors have considered the instability in duct flows with conducting Hartmann walls (Reed and Picologlou, 1989; Kinet et al., 2009). These flows however are characterized by strong side wall jets, and undergo a completely different transition; they are not directly relevant to the present work.

It is clear that the experiment of Moresco and Alboussière (2004) is far from fully understood. Currently, it is unfortunately not possible to access the whole parameter range covered in the experiment with numerical simulations. Moreover, even the laminar behavior of MHD toroidal duct flow is relatively unexplored. To the best of our knowledge, only two studies of the laminar flow in such a configuration have been undertaken. The first one was performed by Baylis and Hunt (1971). They used an asymptotic approach, i.e. they assumed the existence of an inertialess and inviscid core, which is surrounded by thin shear layers. Their results show that the inertial term is negligible when the aspect ratio between the length and the average radius R of the duct is small compared to M/R. One decade later, Tabeling and Chabrerie (1981) performed a more detailed analysis, in which they considered the curvature as a small parameter. This allowed them to compute the secondary flow profile in the shear layers for sufficiently low values of the curvature and the Reynolds number and high values of the Hartmann number. They predicted that streamwise-oriented vortices would occur in the parallel layers, whose exact shape depends on the electric boundary conditions. Furthermore, they found that the secondary flow has a strong radially inward component along the Hartmann layers.

Hence, the aim of this work is to compute the full solution the MHD flow of a liquid metal in a toroidal duct. Our work is organized as follows: first, we outline the mathematical formulation and the computational details of our simulations. The two following sections are devoted to a discussion of the results for respectively steady and non-steady flows. The last section summarizes the most important conclusions of this work.

2. Mathematical model and computational method

We consider the incompressible flow, characterized by a velocity field $\mathbf{u}$, of a fluid in a square annular duct with mean radius R and length $2L$ (see Fig. 1). The axis of the torus is along the y-direction. The material properties of the fluid, like its mass density ρ, kinematic viscosity ν and electrical conductivity σ are assumed to be constant. The flow is subjected to a uniform magnetic field $\mathbf{B} = B_0\mathbf{1}_y$. Moreover, we assume that the magnetic Reynolds $R_m \ll 1$, so that the magnetic field does not change with time; this is called the quasi-static approximation. This means that the induced electric field can be derived from a scalar potential function ϕ (Roberts, 1967). The electric current density $\mathbf{j}$ obeys Ohm's law for a moving conductor:

$$\mathbf{j} = \sigma(-\nabla\phi + \mathbf{u} \times \mathbf{B}) \tag{1}$$

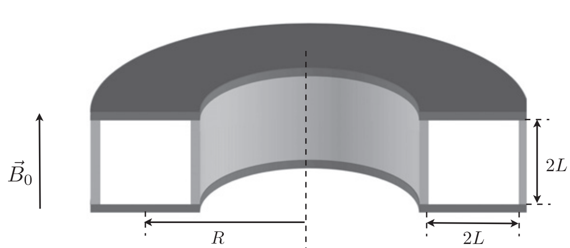

Fig. 1. Sketch of the annular geometry. The color of the walls indicates their electric conductivity: perfectly conducting (light gray), and perfectly insulating (dark gray).

The constraint of charge conservation under the quasi-neutrality assumption, $\nabla \cdot \mathbf{j} = 0$, leads to a Poisson equation for the potential:

$$\nabla^2\phi = \nabla \cdot (\mathbf{u} \times \mathbf{B}) \tag{2}$$

The equations of mass and momentum conservation are the standard incompressible Navier–Stokes equations in which a Lorentz force term is added:

$$\rho(\partial_t\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u}) = -\nabla p + \rho\nu\nabla^2\mathbf{u} + \mathbf{j} \times \mathbf{B} \tag{3}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{4}$$

The boundary conditions for $\mathbf{u}$ and ϕ are the following. For the velocity, we apply standard no-slip conditions on all the walls. The electrical boundary conditions are inspired by the work of Moresco and Alboussière (2004). This means that we have perfectly insulating Hartmann walls and perfectly conducting side walls. Mathematically:

$$\mathbf{u} = 0, \quad \partial_n\phi = 0 \text{ at } y = \pm L \tag{5}$$

$$\mathbf{u} = 0, \quad \phi = \pm V/2 \text{ at } r = R \pm L \tag{6}$$

This makes clear why we do not need an external forcing term in Eq. (3). By imposing a voltage difference between the side walls, a radial current is injected in the fluid. The Lorentz force resulting from the interaction between this current and the magnetic field, provides the necessary forcing of the flow. Our formulation is slightly different from the one of Moresco and Alboussière (2004) in the sense that, in the experiment, the amount of injected current at the side walls was fixed, rather then the potential. In other words: The side wall potential in the experiment was thus constant in space, but still varying in time, while we allow fluctuations in time of the total amount of injected current. In the laminar regime, both formulations are of course strictly equivalent.

We can use the linearity of the Laplacian operator to split Eq. (2) with boundary conditions (5), (6) in two parts: $\phi = \phi_1 + \phi_2$. Here, ϕ_1 is a solution of the non-homogeneous Eq. (2) with Neumann conditions on the Hartmann walls and homogeneous Dirichlet conditions $\phi_1 = 0$ on the side walls. ϕ_2 on the other hand is a solution of the Laplace equation $\nabla^2\phi_2 = 0$, also with Neumann conditions, but now with non-homogeneous Dirichlet conditions $\phi_2 = \pm V/2$ at $r = R \pm L$. In the present geometry, the potential ϕ_2 and the corresponding external forcing $\mathbf{f}_{ext}$ take the following form:

$$\phi_2 = \frac{V}{\ln\left(\frac{R+L}{R-L}\right)} \ln\left(\frac{r}{\sqrt{R^2 - L^2}}\right) \tag{7}$$

$$\mathbf{f}_{ext} = -\sigma\nabla\phi_2 \times \mathbf{B} = f_{ext}\mathbf{1}_\theta = \frac{VB}{r\ln\left(\frac{R+L}{R-L}\right)}\mathbf{1}_\theta \tag{8}$$

We see that $\nabla\phi_2$, and thus the external forcing, decrease as $1/r$ in radial direction and are independent of the velocity field.

The different cases that we will consider can be characterized by three non-dimensional numbers: the well-known Reynolds number Re, the Hartmann number M, and the ratio between the duct length and the mean radius of the annulus:

$$Re = \frac{UL}{\nu} \tag{9}$$

$$M = B_0L\sqrt{\frac{\sigma}{\rho\nu}} \tag{10}$$

$$\zeta = \frac{L}{R} \tag{11}$$

In the definition of Re, the characteristic velocity U is defined as the bulk streamwise velocity. We use a finite-volume method to discretize the equations. Our code is called YALES2, and is discussed in Moureau et al. (2011). All the variables are defined at the centers of the control volumes. It is however necessary to define velocities at the control volume faces to avoid spurious pressure oscillations.