International Journal of Heat and Mass Transfer 100 (2016) 627–633

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

# Temperature dependent thermal conductivity determination and source identification for nonlinear heat conduction by means of the Trefftz and homotopy perturbation methods

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ARTICLE INFO

*Article history:*
Received 14 February 2016
Received in revised form 8 April 2016
Accepted 18 April 2016
Available online 14 May 2016

*Keywords:*
Nonlinear heat conduction
Trefftz method
Coefficient inverse problem
Source identification
Homotopy perturbation method

ABSTRACT

The homotopy perturbation method (HPM) combined with Trefftz method is employed to find the solution of two kinds of nonlinear inverse problems for heat conduction. The first one is a coefficient inverse problem. The thermal conductivity coefficient, with the prescribed boundary conditions on a part of the boundary and with some measured or anticipated values of the solution in some inner points, is determined. The thermal conductivity is assumed to have a form of a linear function of temperature with unknown coefficients. The number of T-functions in HPM is chosen to obtain the best fitting of the approximate solution to the input data. Minimization of the difference between the input data and the approximate solution of the problem, leads to the values of coefficients describing in an approximate way the unknown coefficients. The second problem is determination of an unknown source term for nonlinear stationary heat conduction with prescribed boundary conditions and some measured or anticipated values of the solution in some inner points. The source term is assumed to have a form of a polynomial with unknown coefficients. Number of the coefficients determines the number of functions in HPM resulting from expansion of $H(v,p)$ with respect to the parameter $p$ in order to find the components of the approximate solution of the problem. The components consist of Trefftz functions for the linear parts of the resultant equations based on powers of p-terms. Minimization of the difference between the values prescribed or measured inside the considered domain and the approximate solution of the problem, leads to the values of coefficients describing in an approximate way the source term.



## 1. Introduction

Identification of the thermal conductivity coefficient and heat source is important in engineering practice. Both problems belong to the class of inverse problems and can be solved by means of homotopy perturbation method combined with Trefftz method. The homotopy perturbation method (HPM) has been worked out over a number of years by numerous authors, mainly by He, [1] and many others. Basic idea of HPM is presented in the next section. Thanks to HPM the nonlinear problems can be reduced to the series of linear problems. The main idea proposed in this paper is solving of the linear problems by means of Trefftz method at any step of HPM. Basics of the Trefftz method is described in the paper [2]. This method is addressed to partial differential equations and has been. The solution is approximated by linear combination of the functions which satisfy the given equation identically. So, the approximate solution also satisfies the equation identically. Initial and boundary conditions are fulfilled only approximately. A lot of scientists have been developed this method for different equations and problems. Nowadays the Trefftz method has a lot of streams. The approach presented in this paper is widely described for linear heat conduction equation in [3–6], for wave equation in [7], for thermo-elasticity problems in [8], for plate vibration in [9] and for beam vibration in [10]. The Trefftz method is suitable for solving different types of inverse problems, such as the boundary inverse problems [3,6–8,11–13], source identification [9,10,14], coefficient inverse problems [12,15] and backward heat conduction problems [4,16].

Problem of thermal conductivity determination has been investigated in numerous papers. In [17] identification of temperature dependent conductivity was considered with iterative process used to solve the problem. To solve similar problem, in [18] used Levenberg–Marquardt method. Also in [19] this type of problem

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http://dx.doi.org/10.1016/j.ijheatmasstransfer.2016.04.103
0017-9310/

**Nomenclature**

| | | | |
|---|---|---|---|
| $D$ | domain | $k$ | heat conduction coefficient |
| $\Gamma$ | boundary of $D$ | $Q$ | inhomogeneity |
| $x,\ y$ | spatial variables | $g_j$ | $j$th Trefftz function |
| $t$ | time | $\rho$ | mass density |
| $A$ | partial differential operator | $c$ | heat capacity |
| $f$ | source | $l$ | length |
| $L$ | linear partial differential operator | $W$ | number of thermocouples |
| $N$ | nonlinear partial differential operator | $H$ | number of measurements |
| $H$ | homotopy | $c_i$ | coefficients |
| $u,\ v$ | functions | $F$ | functional |
| $p$ | parameter | | |

was investigated applying the gradient method coupled with the BEM. To determine conductivity depending on spatial variable Liu [20] used a Lie-group shooting method, and in [21] the BEM-Based Genetic Algorithm was applied. In [22] an inverse boundary element method was developed to characterize the spatially dependent conductivity $k(x)$ of heterogeneous materials. The effective method of thermal conductivity identification is presented in [23], but comparing to the approach presented in this paper is more complicated and leads to discrete description of the identified coefficients, including the coefficient of heat conduction.

On the other hand the problem of heat source identification has been investigated in [24] with the use of Laplace transform, in [25] using MES, in [26] applying MFS and Laplace transform, in [27] (strengths of the distinct point sources), [28] (BEM for time-dependent source), [29] (a regularization procedure, based on the mollification method, and a marching scheme for the solution of the stabilized problem), [30] (using Green's function) and in many other papers. However, in the problems of thermal conductivity determination as well as in the problems of heat source identification neither Trefftz method nor HPM was not applied.

In the second section of the paper the method of solving nonlinear problems of heat conduction is described. The method of thermal conductivity identification is presented in the third section. In fourth section the method of source determination is described.

## 2. Basic idea of the homotopy perturbation method

Let us consider a PDE

$$A(u) - f(r) = 0, \quad r \in D \tag{1}$$

with corresponding boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \partial D. \tag{2}$$

Here $A,\ B,\ f(r)$ and $r$ are a differential operator, a boundary operator, an analytical function (called a source term) and a point in a domain $D$, respectively. The differential operator $A$ can be nonlinear; in such a case it can be divided into two parts: $L$ (a linear part) and $N$ (a nonlinear part). Therefore, Eq. (1) can be written as

$$L(u) + N(u) - f(r) = 0, \quad r \in D. \tag{3}$$

Considering the homotopy technique, a homotopy $v(r,p) : D \times [0,1] \to R$ can be constructed as follows [1]

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5}$$

where $r \in D,\ p \in [0,1]$ is an embedding parameter and $u_0$ is an initial approximation for a solution of Eq. (1), which satisfies the boundary conditions, Eq. (2). Obviously, from Eqs. (4) and (5) we find

$$H(v,0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{7}$$

and the changing process of $p$ from zero to unity is just that of $H(v,p)$ from $L(v) - L(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation, and $A(v) - f(r)$ are called homotopic, [31].

The solution of Eq. (4), $v(r,p)$, can be expanded into a power series with respect to the parameter p

$$v(r,p) = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots. \tag{8}$$

Setting $p = 1$ in Eq. (8) results in the solution of Eq. (1)

$$u(r) = \lim_{p \to 1} v(r,p) = v_0 + v_1 + v_2 + v_3 + \ldots. \tag{9}$$

The approximate solution of Eq. (1) can be obtained in a form of truncated series

$$u_N(r) = \sum_{n=0}^{N} v_n. \tag{10}$$

The convergence of this method has been proved by He in [1,32]. The convergence rate depends on the operator $A(v)$, [33].

In the further consideration the initial approximation is chosen in arbitrary way. Then, bearing in mind that the solution, (9), has to satisfy both, the Eq. (1) and boundary conditions (2), conditions for the solution of equations resulting from the fact that the coefficients of the successive powers of a parameter $p$ must be equal to zero, have to be constructed in such way, that the boundary conditions for the solution (9) have to be finally in the form of Eq. (2).

## 3. Thermal conductivity identification

In the range of temperature from 300 K to 600 K the heat conduction coefficient for many solids can be approximated with acceptable accuracy by a linear function of temperature, $u$. For instance for copper $k(u) = 0.0696u + 421.19$, for carbon steel AISI1010 $k(u) = 0.0476u + 77.922$, for stainless steel AISI302 $k(u) = 0.0141x + 11.363$ (for numerical data see Mills, 1999). In the presented paper both, the Trefftz method and the HPM are employed in order to identify the heat conduction coefficient depending linearly on temperature as well as to find approximate solution of the heat conduction problem.

Assume that the heat conduction coefficient, $k(u)$, is an unknown function of $u$, standing for temperature. Instead, some anticipated or measured values of the temperature $u(x,t)$, are known inside the time–space domain (so called internal responses, abbr. IRs),