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Uncertainty propagation of heat conduction problem with multiple random inputs

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ABSTRACT

Uncertainty propagation analysis in engineering systems constitutes a significant challenge. To effectively solve the uncertain heat conduction problem with multiple random inputs, a random collocation method (RCM) and a modified random collocation method (MRCM) are established based on the spectral analysis theory. In both methods, the truncated high-order polynomial series is adopted to approximate the temperature responses with respect to random parameters, and the eventual probabilistic moments are derived by using the orthogonal relationship of polynomial bases. In the pivotal process of calculating the expansion coefficients, RCM evaluates the deterministic solutions directly on full tensor product grids, whereas the Smolyak sparse grids are reconstructed in MRCM to avoid the huge computational cost caused by high dimensions. Comparing the results with traditional Monte Carlo simulation, two numerical examples verify the remarkable accuracy and effectiveness of the proposed methods for random temperature field prediction in engineering.

1. Introduction

In recent years, numerical heat transfer with given deterministic parameters has undergone a rapid development [1]. However, the ubiquitous system variability due to aggressive environment factors, incomplete knowledge and inevitable measurement errors makes the nondeterministic methods more feasible in practical engineering [2]. The main techniques to characterize system uncertainties can be grouped into three categories: probabilistic method, fuzzy theory and interval analysis [3–5]. In most cases, the input uncertainties can be modeled as random variables with probability density functions, which means the probabilistic method can be considered as the most valuable strategy for uncertainty propagation analysis [6,7].

It is a common practice in engineering to use mean values for uncertain parameters and then introduce the safety factors to over specify the system responses [8]. This treatment is easy to implement, but it cannot satisfy the requirement for elaborate analysis of the modern engineering systems. Thus, the probabilistic modeling and analyzing methods have gained an increasing popularity resorting to the reliable numerical results when the probabilistic characteristics of input parameters (such as probability density distribution, mean value, variance, etc.) are given. Up to now, Monte Carlo simulation is still considered as the simplest approach to solve uncertain problems in a probabilistic framework [9,10], but its accuracy strongly depends on the sample number. Considering the excessive computational cost, Monte Carlo method is commonly introduced as a referenced approach, rarely used in practical engineering. The random moment approach, whose objective is to calculate the moments of system responses directly needs to construct a closed equation system by assuming that a higher-order term is a function of the lower-order terms [11,12]. However, there is no general strategy for solving the closure problem. Besides, the error caused by most closure techniques is not easy to quantify. The random perturbation method, based on the Taylor series of random quantities around their mean values, has been widely used in practice due to its smaller computational cost and easily guaranteed convergence condition [13,14]. But this method often requires the uncertainty level of random parameters to be small enough because of the finite-term truncation. For the popular engineering problems with large uncertainties, the results obtained by perturbation method will be unacceptable.

In recent years, the spectral method has become another effective approach for the random uncertainty propagation [15]. Based on the polynomial chaos theory firstly introduced by Ghanem [16], Xiu and Karniadakis presented a developed spectral decomposition method for the transient heat conduction problem subjected to random inputs [17]. It should be noted that besides selecting appropriate polynomial type, the main work of spectral method

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is to calculate expansion coefficients in polynomial series. Up to now, there are two kinds of approaches for the expansion coefficient calculation. The first one is known as the Galerkin approach, where the residue of random governing equations is orthogonal to the linear space spanned by polynomials [18]. By using the Galerkin projection, Xiu and Shen derived a set of deterministic equations for the expansion coefficients in random diffusion problem [19]. Pettersson et al. adopted the Galerkin approach to investigate the Euler equations subjected to random uncertainties in initial and boundary conditions [20]. Although the accuracy of Galerkin approach is optimal, new codes are needed to solve a large number of coupled equations, which can be considered as its main disadvantage, especially for the problems with complex and nonlinear forms. The second one is known as the collocation method, where only a set of deterministic codes need to be run at the selected nodes [21]. Based on the tensor product of Gauss points, Babuska et al. proposed a collocation method for solving elliptic partial differential equations with random coefficients and forcing terms [22]. Using the tensor product Lagrange polynomials, Bressolette et al. applied the collocation method to analyze large classes of uncertain mechanical problems with random input data [23]. To avoid the huge computational cost caused by the high-dimensional full tensor product grids, some research work has been done on sparse grids [24]. Compared with Galerkin approach, the main advantage of collocation method is ease of implementation. Nevertheless, current research on random collocation method is mainly concentrated in the special fields of mathematics and structural mechanics, while its application in uncertain heat transfer problems is mostly unexplored.

In this study, it is the first time that the collocation methods have been used to solve uncertain heat conduction problem in engineering. Besides, by combining the nested Clenshaw–Curtis nodal sets with Smolyak algorithm, the collocation method is further improved. The paper is structured as follows. The basic theory of polynomial approximation for random heat conduction problem is introduced in Section 2. Then two random collocation methods are presented in the next two sections. The first one is RCM where the Clenshaw–Curtis collocation points are directly constructed by the full grids. To improve the computational efficiency for the high-dimensional problems, the Smolyak algorithm is adopted in MRCM to reconstruct the sparse grids. In Section 5, two numerical examples about a thermal plate and a sandwich structure are provided to demonstrate the effectiveness and accuracy of proposed methods, and we conclude the paper with a brief discussion at last.

2. Polynomial approximation of random temperature response

The governing equation of steady heat conduction problem with a heat source can be expressed as

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial T(x)}{\partial x}\right)+f(x)=0 \quad x\in\Omega \tag{1}$$

where Ω is a bounded domain; $T(x)$ denotes the temperature response; $k(x)$ stands for the heat conductivity, and $f(x)$ is the intensity of heat source.

For the interior domain Ω bounded by Γ as shown in Fig. 1, four kinds of boundary conditions are considered as follows

$$\begin{aligned} &T|_{\Gamma_1}=T_s\\ &-k\frac{\partial T}{\partial \mathbf{n}}\Big|_{\Gamma_2}=q_s\\ &-k\frac{\partial T}{\partial \mathbf{n}}\Big|_{\Gamma_3}=h(T-T_e)\\ &-k\frac{\partial T}{\partial \mathbf{n}}\Big|_{\Gamma_4}=\sigma\varepsilon(T^4-T_e^4)\end{aligned} \tag{2}$$

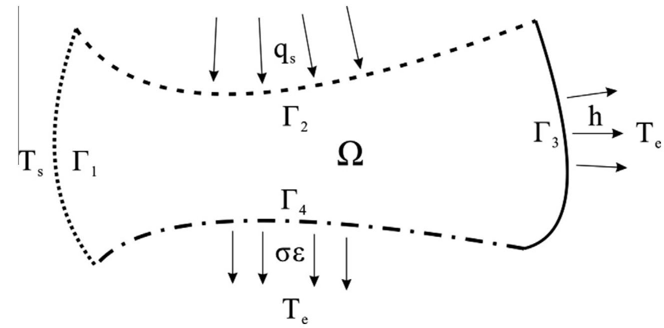

Fig. 1. Four kinds of boundary conditions.

where T_s is the given boundary temperature; $\mathbf{n}$ is the normal vector; q_s denotes the boundary heat flux; h, T_e represent the heat transfer coefficient and ambient temperature; σ, ε stand for the Stefan–Boltzmann constant and surface emissivity.

For the practical heat conduction problem in engineering, due to vaguely defined system characteristics and insufficient information, the uncertainties in material properties, external loads and boundary conditions are unavoidable. In this paper, the uncertain input parameters whose probability distribution functions can be defined unambiguously based on the sufficient experiment data are expressed as a random vector $\xi=(\xi_1,\xi_2,\ldots,\xi_n)$, where n denotes the number of random variables. Therefore, the governing Eq. (1) with multiple random inputs can be rewritten as

$$\frac{\partial}{\partial x}\left(k(x;\xi)\frac{\partial T(x;\xi)}{\partial x}\right)+f(x;\xi)=0 \quad (x;\xi)\in\Omega\times \mathrm{V} \tag{3}$$

where V stands for the probability space.

As we know, the traditional probabilistic methods based on Taylor series expansion only adopt the limited information around the mean value to approximate the response function, whose accuracy becomes unacceptable if the uncertainty level of random parameters is not small enough. To overcome this shortcoming, in this paper we will adopt the high-order polynomial series to approximate the temperature response with a much higher accuracy level. According to the framework of polynomial chaos, the random temperature response $T(x;\xi)$ can be expanded as

$$T(x;\xi)=\sum_{\mathbf{i}}T_{\mathbf{i}}(x)\Phi_{\mathbf{i}}(\xi) \tag{4}$$

where $\Phi_{\mathbf{i}}(\xi)$ stands for the n-dimensional orthogonal polynomial basis, and can be determined in advance based on the distribution type of random variables; $T_{\mathbf{i}}(x)$ represents the corresponding expansion coefficient, and $\mathbf{i}=(i_1,i_2,\ldots,i_n)$ denotes the multi-index. In practice, a maximum order N is often selected to truncate the polynomial series up to finite terms, by which Eq. (4) can be concisely rewritten as

$$T(x;\xi)\approx T_N(x;\xi)=\sum_{|\mathbf{i}|\leqslant N}T_{\mathbf{i}}(x)\Phi_{\mathbf{i}}(\xi) \tag{5}$$

where $|\mathbf{i}|=i_1+i_2+\cdots+i_n$, and the total number of polynomial expansion terms can be calculated by C_{n+N}^{n} with respect to the number of variables n and the order of polynomial series N.

By using the orthogonal relationship of polynomial bases $\langle\Phi_{\mathbf{i}}\Phi_{\mathbf{j}}\rangle=\gamma_{\mathbf{i}}\delta_{\mathbf{ij}}$, where $\gamma_{\mathbf{i}}=\gamma_{i_1}\cdots\gamma_{i_n}$ is the normalization factor and $\delta_{\mathbf{ij}}$ is the n-variate Kronecker delta function, the expectation and variance of $T(x;\xi)$ can be approximated by

$$E[T(x;\xi)]\approx E[T_N(x;\xi)]=T_0(x) \tag{6}$$

$$Var[T(x;\xi)]\approx Var[T_N(x;\xi)]=\sum_{0<|\mathbf{i}|\leqslant N}\left(\gamma_{\mathbf{i}}T_{\mathbf{i}}^2(x)\right) \tag{7}$$