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The modified polynomial expansion method for identifying the time dependent heat source in two-dimensional heat conduction problems

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ABSTRACT

In this paper, the modified polynomial expansion method is developed to solve problems of identifying the time dependent heat source, in which an inverse problem is encountered. Aimed at this problem, the variation of variables is adopted to eliminate the unknown heat source and obtain a six-line boundary value problem. As compared with the conventional four-line boundary value problem, the six-line boundary value problem is quite hard to be dealt with. After the unknown non-homogeneous term being eliminated, the polynomial expansion method is introduced to discretize the time and space fields, respectively. Then, the distribution of temperature is expressed as a linear superposition of polynomial functions. After that, a characteristic length concept is adopted to resolve the ill-posed matrix problems arising in those conventional polynomial expansion methods. The desired heat source function can be obtained by putting the solution of the six-line boundary value problem into differential operations. Several numerical experiments with designed examples are included to validate the accuracy and effectiveness of the proposed approach.

1. Introduction

The inverse heat source problems are solving the heat conduction equation with unknown heat source. It is very difficult to identify the heat source due to the ill-posedness of the inverse problem in nature [1,2]. Since Stolz [3] first gave a numerical solution of the inverse heat conduction problem, there have been many applications in different scientific and engineering fields after his work. There are also various types of inverse problems for heat conduction equation. For example, Masood and Yousuf [4] used the finite difference method to reconstruct the initial temperature distribution; Vakili and Gadala [5,6] estimated the missing boundary temperature or heat flux by using the particle swarm optimization; Tan and Liu [7] used the collocation meshless method to solve the inverse geometry design problem; Molavi et al. [8] used the modified Levenberg–Marquardt method to identify the thermal parameters. In this paper, we focus on the inverse heat source problem aiming to recover the unknown time dependent heat source.

There were many studies to identify different types of heat sources. Cannon and Duchateau [9] estimated the nonlinear temperature dependent heat source. Savateev and Duchateau [10] and Borukhov and Vabishchevich [11] recovered the heat sources, which are functions of space and time, and the heat sources are additive or separable on space and time axes; however, they only gave a priori estimate of numerical error in [11]. The method of fundamental solutions (MFSs) can solve the homogeneous heat conduction problems. Mierzwiczak and Kołodziej [12] applied the MFS associated with Laplace transformation to identify the heat source, which is function of space and time. Some investigators reconstructed the time- or space-dependent heat source, such as Farcas and Lesnic [13], Ling et al. [14], Shi et al. [15], Yan et al. [16], and Yang et al. [17], and the regularization technique was introduced to stabilize their algorithms.

For solving an inverse problem, we usually need extra measurement data. To recover the time-dependent heat source, we choose some measure points and record the temperature history at these points. Traditionally, the problem can be reduced into a Volterra integral equation of the first kind; however, it is also ill-posed. The numerical solution can be obtained by solving the above integral equation with some regularization techniques. Maalek Ghaini [18] proved the existence, uniqueness and stability of this problem, but he did not give numerical procedures and demonstrated examples.

There are also many researches about reconstructing the time-dependent source. Huang and Shih [19] estimated the forcing

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Nomenclature

u	temperature	m	degree of polynomials
H	heat source	n	number of points
t	temporal coordinate		
x, y	spatial coordinate	*Subscripts*	
t	initial condition	i, j, k	indices
F, G	boundary condition		
T	transformed temperature	*Superscripts*	
$\mathbf{A}$	stiffness matrix of weighting coefficients	g	governing equation
$\mathbf{c}$	vector of weighting coefficients	0	initial condition
$\mathbf{b}$	vector of known data	b	boundary condition
a, d	element of weighting coefficients		

term of the Euler–Bernoulli beam. Marin [20] presented the MFS to identify the heat source of the heat conduction equation in the steady state. Marin et al. [21] proposed the boundary element method to solve the inverse problem in magnetic resonance imaging gradient coils. Mera et al. [22] detected the super-elliptical inclusions that are applied in tomography. Wen [23] used the MFS associated with the Tikhonov regularization method to estimate the heat source and partial initial temperature. Liu [24] reconstructed a past-time dependent heat source by the Lie-group shooting method. Yeih and Liu [25] constructed a three-point boundary value problem and solved the resultant system by a two-stage Lie-group shooting method. Liu [26] proposed a self-adaptive Lie-group shooting method to recover the initial condition or the heat source. In his approach, no extra measurement data are needed. Yang et al. [27] transferred the inverse heat source problem into an optimal control problem. Recently, Kuo et al. [28] have proposed a modified polynomial expansion method to solve the one-dimensional inverse heat source problem. By considering the characteristic length, the proposed method can reach a convergent solution and greatly improve the stability of the algorithm.

In the literature reviewed above, we can see that many investigators solved the inverse heat source problem by using various approaches, but there are still stability and multi-dimensional problems to be overcome. Some researchers assumed the function type with some unknown coefficients to obtain the unknown heat source function and the unknown coefficients can be yielded by the minimization technique, but the function type would influence the estimating results. In addition, some researchers might transform the problems into an integral equation; however, it is still difficult to be solved due to the ill-posedness of the integral equation. Besides, some investigators introduced the variable transformation and obtained a three-point boundary value problem, but it is difficult to solve the resultant problem by the traditional mesh-based methods. The Tikhonov regularization technique is usually used to reduce the ill-posedness of the inverse problems, but it needs extra computation to choose the regularization parameter by using L-curve or iteration methods, which is time consuming.

The concept of the characteristic length for series solution was first proposed by Liu [29,30]. In his papers, the conventional Trefftz basis functions are scaled by a factor which is the so-called characteristic length and plays a major role in reducing the condition number of the resultant linear system and improving the accuracy for both the direct and indirect Trefftz methods. We can find that the basis functions in the polar coordinate can be transformed into the polynomial functions in the Cartesian coordinate. Therefore, the basis functions for Laplace equation can be considered as a special polynomial expansion with the characteristic length. Later, Liu and Atluri [31] solved the interpolation problems by using the characteristic length. Liu [32] also proposed a polynomial expansion method with multi-scale characteristic length for solving the interpolation problems.

In this paper, we develop a modified polynomial expansion method to recover the time dependent heat source together with the temperature distribution in the two-dimensional heat conduction equation. The present approach is efficient because the characteristic length implies the regularization. Thus, we can select the value easily without extra computation. Several numerical examples will be provided to validate the present approach and demonstrate the accuracy and stability of the present method.

2. Six-line boundary value problem

Let us consider a two-dimensional heat conduction problem:

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + H(t), \quad 0 < x < l_x, \quad 0 < y < l_y, \quad 0 < t \leqslant t_f, \tag{1}$$

$$u(x,y,0) = f(x,y), \tag{2}$$

$$u(0,y,t) = F_0(y,t), \quad u(l_x,y,t) = F_l(y,t), \quad u(x,0,t) = G_0(x,t), \quad u(x,l_y,t) = G_l(x,t), \tag{3}$$

where $H(t)$ denotes the time dependent heat source, $f(x, y)$ denotes the initial temperature distribution, $F_0(y, t)$, $F_l(y, t)$, $G_0(x, t)$ and $G_l(x, t)$ are the left, right, bottom, and top boundary conditions of the rectangular domain, respectively. To identify the unknown heat source, we give overspecified temperature data:

$$u(x_m,y,t) = F_m(y,t), \quad u(x,y_m,t) = G_m(x,t). \tag{4}$$

Yan et al. [16] proposed a variable transformation for the one-dimensional inverse heat source problems:

$$T(x,t) = u(x,t) - \int_0^t H(s)ds. \tag{5}$$

We applied the same transformation into the two-dimensional problem by $T(x,y,t) = u(x,y,t) - \int_0^t H(s)ds$ and Eqs. (1)–(4) are transformed into a homogeneous PDE:

$$\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial^2 T(x,y,t)}{\partial x^2} + \frac{\partial^2 T(x,y,t)}{\partial y^2}, \quad 0 < x < l_x, \quad 0 < y < l_y, \quad 0 < t \leqslant t_f, \tag{6}$$

$$T(x,y,0) = f(x,y), \tag{7}$$

$$T(x_m,y,t) - T(0,y,t) = F_m(y,t) - F_0(y,t), \quad T(l_x,y,t) - T(x_m,y,t) = F_l(y,t) - F_m(y,t), \tag{8}$$