International Journal of Heat and Mass Transfer 95 (2016) 936–943

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

The multiple-scale polynomial Trefftz method for solving inverse heat conduction problems

Chein-Shan Liu [a,b], Chung-Lun Kuo [a,b,*], Wun-Sin Jhao [b]

[a] Center for Numerical Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing, Jiangsu 210098, China
[b] Department of Civil Engineering, National Taiwan University, Taipei 106-17, Taiwan

ARTICLE INFO

Article history:
Received 24 August 2015
Received in revised form 30 December 2015
Accepted 2 January 2016
Available online 13 January 2016

Keywords:
Inverse problem
Backward heat conduction problem
Inverse heat source problem
Heat polynomials
Multiple-scale polynomial Trefftz method

ABSTRACT

The polynomial Trefftz method consists of the polynomial type solutions as bases, providing a cheap boundary-type meshless method to solve the heat conduction equation, since the bases automatically satisfy the governing equation. In order to stably solve the backward heat conduction problem (BHCP), and the inverse heat source problem (IHSP) together with the boundary condition recovery problem by a polynomial Trefftz method, which are both known to be highly ill-posed, we introduce a multiple-scale post-conditioner in the resultant linear system to reduce the condition number. Then the conjugate gradient method (CGM) is used to solve the post-conditioned linear system to determine the unknown expansion coefficients. In the multiple-scale polynomial Trefftz method (MSPTM) the scales are determined a priori by the collocation points on space–time boundary, which can retrieve the missing initial data, the unknown time-dependent heat source as well as the boundary condition rather well. Several numerical examples of the inverse heat conduction problems demonstrate that the MSPTM is effective and accurate, even for those of severely ill-posed inverse problems under very large noises.

1. Introduction

Let us consider a one-dimensional heat conduction equation:

$$u_t(x,t) = \alpha u_{xx}(x,t), \quad (x,t) \in \Omega := \{0 < x < \ell, 0 < t \leqslant t_f\}, \tag{1}$$

where the subscripts x and t denote the partial differentials with respect to x and t, respectively. For the direct problem we specify boundary conditions and initial condition on $\Gamma = \{x = 0, 0 \leqslant t \leqslant t_f\} \cup \{x = \ell, 0 \leqslant t \leqslant t_f\} \cup \{0 \leqslant x \leqslant \ell, t = 0\}$. In contrast, for the backward problem we specify boundary conditions and final time condition on $\Gamma = \{x = 0, 0 \leqslant t \leqslant t_f\} \cup \{x = \ell, 0 \leqslant t \leqslant t_f\} \cup \{0 \leqslant x \leqslant \ell, t = t_f\}$.

Suppose that the solution of Eq. (1) is in the following form:

$$u(x,t) = \exp(\gamma x + \gamma^2 \alpha t), \tag{2}$$

where γ is a small parameter, which can be expanded in terms of γ by using the Taylor series:

$$u(x,t) = \sum_{n=0}^{\infty} p_n(x,t) \frac{\gamma^n}{n!}. \tag{3}$$

On the other hand we have

$$\exp(\gamma x) = \sum_{n=0}^{\infty} \frac{\gamma^n x^n}{n!}, \tag{4}$$

$$\exp(\gamma^2 \alpha t) = \sum_{k=0}^{\infty} \frac{\alpha^k t^k \gamma^n}{k!}, \quad n = 2k. \tag{5}$$

Inserting them into Eq. (2) and equating it to Eq. (3) we can derive

$$p_n(x,t) = n! \sum_{k=0}^{[n/2]} \frac{\alpha^k t^k x^{n-2k}}{k!(n-2k)!}, \tag{6}$$

which is known as the heat polynomial, and can be verified satisfying Eq. (1) automatically as follows:

$$\frac{\partial p_n(x,t)}{\partial t} = n! \sum_{k=0}^{[n/2]} \frac{k\alpha^k t^{k-1} x^{n-2k}}{k!(n-2k)!} = n! \sum_{k=0}^{[n/2]} \frac{(k+1)\alpha^{k+1} t^k x^{n-2k-2}}{(k+1)!(n-2k-2)!} = n! \sum_{k=0}^{[n/2]} \frac{\alpha^{k+1} t^k x^{n-2k-2}}{k!(n-2k-2)!},$$

$$\frac{\partial^2 p_n(x,t)}{\partial x^2} = n! \sum_{k=0}^{[n/2]} \frac{(n-2k)(n-2k-1)\alpha^k t^k x^{n-2k-2}}{k!(n-2k)!} = n! \sum_{k=0}^{[n/2]} \frac{\alpha^k t^k x^{n-2k-2}}{k!(n-2k-2)!}.$$

Upon multiplying the second equation by α, it follows that

$$\frac{\partial p_n(x,t)}{\partial t} = \alpha \frac{\partial^2 p_n(x,t)}{\partial x^2},$$

which is the heat Eq. (1).

* Corresponding author at: Center for Numerical Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing, Jiangsu 210098, China.

E-mail address: eji1215@gmail.com (C.-L. Kuo).

http://dx.doi.org/10.1016/j.ijheatmasstransfer.2016.01.008
0017-9310/

Nomenclature

$\mathbf{A}$	coefficient matrix in Eq. (8)
$\mathbf{a}_i$	ith column of $\mathbf{A}$
$\mathbf{b}$	the right-hand side in Eq. (8)
$\mathbf{b}_1$	$:= \mathbf{A}^{\mathrm{T}}\mathbf{b}$
c_i	expansion coefficients
$\mathbf{c}$	n-dimensional vector of coefficients
$\mathbf{D}$	$:= \mathbf{A}^{\mathrm{T}}\mathbf{A}$
$g(x)$	initial value of u
$g\prime(x)$	initial value of v
$h(t)$	right-boundary value of u
$H(t)$	time-dependent heat source
ℓ	length of rod
m	the highest order of heat polynomials
n	$:= m + 1$
n_c	the number of collocation points
$p_i(x,t)$	heat polynomial
$\mathbf{P}$	post-conditioning matrix
$q_0(t)$	left-boundary value of v
$q_\ell(t)$	right-boundary value of v
$R(i)$	random numbers
s	noise level
s_i	multiple-scale
t	time
t_f	final time
$u(x,t)$	temperature
$u_0(t)$	left-boundary value of u
$u^T(x)$	final time temperature
$v(x,t)$	$:= u_x(x,t)$
x	space variable

Greek symbols

α	heat conductivity in Eq. (1)
Ω	a bounded region
Γ	a boundary
γ	a small parameter
σ	noise level
ε	convergence criterion

Subscripts and superscripts

i	index
k	index
T	transpose

The heat polynomials were introduced by Rosenbloom and Widder [1], and further described by Widder [2–4]. A more comprehensive discussion of the heat polynomial analogies for higher order evolution equations can be seen in [5]. The usage of polynomial expansion as a trial solution of linear partial differential equations (PDEs) is simple and is straightforward to derive the required linear algebraic equations (LAEs) to determine the expansion coefficients after a suitable collocation of points in the problem domain and boundary. However, it is seldom used as a major numerical tool to solve the linear PDEs. The main difficulty is that the resultant LAEs are often highly ill-conditioned. How to reduce the condition number of the linear system is an important issue when one applies the polynomials expansion method to solve the linear PDEs.

As a received heat conduction system, it may be already in the situation of on-line service, and we cannot measure the initial temperature because the initial time is passed. For such a system, although we can exactly know the boundary conditions, but the initial condition is absent, which renders the backward heat conduction problem (BHCP) not easy to be solved. The BHCP is one of the inverse problems for the applications in heat conduction engineering to recover the past history of temperature. The inverse problems are those in which one intends to determine the causes for a desired or observed effect. One of the characterizing properties of many inverse problems is that they are always ill-posed. Mathematically speaking, the linear operator generated from the BHCP is a compact one with an infinite rank, whose inversion is discontinuous, and thus, the solution that continuously depends on the given final time data does not exist.

The numerical schemes adopted for the BHCPs are usually implicit. The explicit ones are apparently not very effective. Mera [6] has mentioned that the BHCP is hard to be solved by using the classical numerical methods and requires special techniques to solve it. In order to solve the BHCP, there appeared certain progresses in this issue, including the boundary element method [7], the iterative boundary element method [8,9], the Tikhonov regularization technique [10,11], the operator-splitting method [12], the lattice-free high-order finite difference method [13], the method of fundamental solutions [6,14,15], the third order mixed-derivative regularization technique [16], the Fourier regularization method [17], the three-spectral regularization methods [18], and the radial-basis functions method [19]. Clark and Oppenheimer [20] and Ames et al. [21] have used a quasi-reversibility method to approximate the solution of the BHCP.

Since 2004 we have developed several methods for solving the BHCP, namely, the group preserving scheme [22], the backward group preserving scheme [23], Lie-group shooting method together with the quasi-boundary regularization [24], the Fredholm integral equation method [25,26], the Lie-group shooting method in the time direction [27], the Lie-group shooting method in the spatial direction [28], the fictitious time integration method [29], a self-adaptive Lie-group shooting method [30], the method of fundamental solutions with conditioning by a new post-conditioner [31], a two-stage group preserving scheme [32], a $GL(n,\mathbb{R})$ shooting method [33], and a Lie-group differential algebraic equations method [34]. Tadi [35] has discussed a different kind of BHCP by imposing two extra heat flux boundary conditions. Recently, Liu [30] has solved the BHCP without resorting on final time condition and overspecified boundary conditions; moreover, Liu [36] can solve the high-dimensional BHCP by using a multiple/scale/direction polynomial Trefftz method.

The remaining portion of the current paper is arranged as follows. In Section 2 we introduce the multiple-scale polynomial Trefftz method by using a derived formula to determine the multiple scales. The problem statement of the BHCP is given in Section 3, whose resulting linear system is solved by using the conjugate gradient method (CGM), and the numerical examples are given in Section 4. Numerical solutions for the simultaneous recovery of the time-dependent heat source and boundary condition are given in Section 5, where we can find that the new method is highly stable and very accurate, although under a large noise 20%. Some conclusions are drawn in Section 6.

2. The multiple-scale polynomial Trefftz method

Let

$$u(x,t) = \sum_{i=1}^{m} c_i p_i(x,t) + c_n, \tag{7}$$