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An analytical solution to the heat transfer problem in Shercliff flow

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ABSTRACT

The study of flow in a rectangular duct, subject to a strong transverse magnetic field is of interest in a number of applications. An important application of such flows is in the context of coolants, where the principle issue of interest is convective heat transfer. For fully developed laminar flows, the problem can be characterised in terms of two coupled partial differential equations. In the case of perfectly electrically insulating boundaries, there is a well known analytical solution due to Shercliff, which provides the velocity and induced magnetic field profiles. In this paper, we demonstrate analytical solutions to H_1 and H_2 heat transfer problems for the Shercliff case in rectangular ducts and obtain temperature profiles and corresponding Nusselt numbers as functions of aspect ratio and Hartmann number.

1. Introduction

The flow in a rectangular duct, subject to strong transverse magnetic fields, is of significant interest in fusion applications due to the use of liquid metal coolants employed in some fusion blanket designs. Depending on the circumstances, this magnetohydrodynamic problem may be simplified by assuming a laminar fully-developed flow with perfectly electrically insulating walls. The problem then reduces to two coupled partial differential equations, whose solution was first obtained by Shercliff [1]. Shercliff obtained explicit analytical solutions for the velocity and magnetic field profiles for this case and his work was subsequently extended to the case of imperfectly and perfectly conducting walls by Hunt [2,3].

In the context of fusion blankets, of equal or greater importance is the concomitant heat transfer, as the extraction of heat is one of the main roles of the blanket itself. Despite the existence of analytical solutions for the velocity profile, there is as yet (to the authors knowledge) no corresponding solution to the heat transfer problem for the Shercliff case (and indeed the Hunt cases). Such solutions exist for flow between parallel plates and flows in circular channels [4,5], and for 1-D heat transfer [6]. There are also some experimental and many numerical studies of heat transfer for Shercliff and related cases [7–12]. It should be noted that even though numerical solutions exist, analytical solutions play an important role in the validation of such computational codes and can give significant insight into the underlying physics, as well as providing approximate parameters for 1-D thermal–hydraulic systems codes. In this article we extend an analytical solution of the temperature profile in rectangular ducts for both the H1 and H2 heat transfer cases, already well developed for the non-MHD case, to the electrically insulating wall MHD case (Shercliff flow). To the author's knowledge this is novel.

2. Problem formulation

Referring to Fig. 1, the momentum equation in a fully developed MHD flow in a rectangular duct of size $-ad_h \leqslant X \leqslant ad_h$ and $-bd_h \leqslant Y \leqslant bd_h$ (where d_h is the hydraulic diameter), subject to an applied X-directed magnetic field B_x^0 is given by

$$\nu\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial Z}+\frac{1}{\rho\mu}\frac{\partial B_z}{\partial X}B_x^0=0 \tag{1}$$

The flow of conducting fluid generates an induced magnetic field B_z, satisfying

$$\frac{1}{\mu\sigma}\left(\frac{\partial^2 B_z}{\partial X^2}+\frac{\partial^2 B_z}{\partial Y^2}\right)+B_x^0\frac{\partial U}{\partial X}=0 \tag{2}$$

where U is the velocity, ν is the kinematic viscosity, μ the magnetic permeability, ρ the density and σ the electrical conductivity of the fluid.

Non-dimensionalising, by setting

$$x=\frac{X}{d_h},\quad y=\frac{Y}{d_h},\quad z=\frac{Z}{d_h} \tag{3}$$

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Nomenclature

Γ	wetted perimeter (m)	Hg	Hagen number
μ	fluid magnetic permeability (Hm^{-1})	Nu	Nusselt number
v	kinematic viscosity ($m^2\,s^{-1}$)	p	pressure (Pa)
ρ	fluid density (kgm^{-3})	Pr	Prandtl number
σ	fluid electrical conductivity (Sm^{-1})	q''	heat flux ($W\,m^{-2}$)
A	duct cross-sectional area (m^2)	Re	Reynolds number
a	duct half-width (m)	T	temperature profile (K)
b	duct half-height (m)	t	dimensionless temperature
B_x^0	applied x-directed magnetic field ($Wb\,m^{-2}$)	T_m	bulk temperature (K)
B_z	induced magnetic field ($Wb\,m^{-2}$)	T_w	wall temperature (K)
d_h	hydraulic diameter (m)	U	velocity profile (ms^{-1})
h	dimensionless magnetic field	u	dimensionless velocity
Ha	Hartmann number	U_m	mean velocity (ms^{-1})

$$u = \frac{U}{U_m} \tag{4}$$

where

$$U_m = \frac{1}{A}\int_A U dA \tag{5}$$

and

$$h = \frac{1}{\mu}\frac{1}{\sqrt{\rho v \sigma}}\frac{1}{U_m}B_z \tag{6}$$

we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Ha\frac{\partial h}{\partial x} = \frac{Hg}{Re} \tag{7}$$

where

$$Ha = B_x^0 d_h \sqrt{\frac{\sigma}{\rho v}} \tag{8}$$

and the Hagen number is defined as

$$Hg = \frac{(\partial p/\partial Z)d_h^3}{\rho v^2} \tag{9}$$

The no-slip condition requires that $u = 0$ at the wall. The induced magnetic field h satisfies

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + Ha\frac{\partial u}{\partial x} = 0 \tag{10}$$

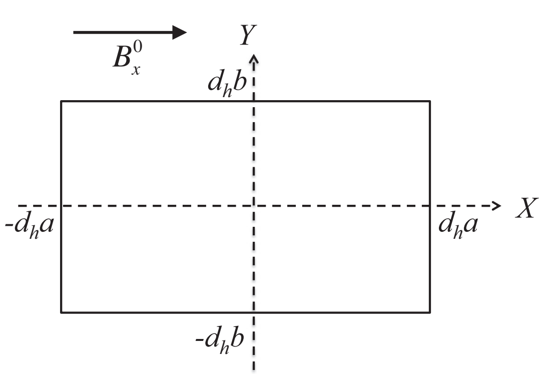

Fig. 1. Duct coordinate system.

in the fluid region. For the Shercliff problem considered here, the induced magnetic field vanishes at the wall. The solution to this problem is well known, and is given in the appendix for the case.

In the following we consider the energy equation, which in steady state, fully developed flow, can be written as

$$\frac{v}{Pr}\left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2}\right) = U\frac{\partial T}{\partial Z} \tag{11}$$

For now, we leave this equation in its dimensional form. The process of non-dimensionalisation differs markedly between H_1 and H_2 cases and is dealt with at the beginning of Sections 3.1 and 3.2 for the H_1 and H_2 cases, respectively.

3. Analytical solution

3.1. H_1 heat transfer case

The H_1 transfer case describes circumstances where the heat flux is uniform in the axial direction and the wall temperature T_w is uniform in the peripheral direction. Under the conditions of fully developed Shercliff flow, it can be assumed that

$$\frac{\partial T}{\partial Z} = \frac{dT_m}{dZ} = const \tag{12}$$

where the bulk temperature T_m is defined as

$$T_m = \frac{\int_A UTdA}{\int_A UdA} \tag{13}$$

We non-dimensionalise as before, with the non-dimensional temperature profile $t(x,y)$ being defined by

$$t = \frac{T}{(dT_w/dZ)d_h} \tag{14}$$

Inserting these into Eq. (11) gives

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = (RePr)u \tag{15}$$

We now proceed to determine the non-dimensional temperature profile $t(x,y)$ by decomposing the solution into a particular integral and a general solution. We obtain the following particular integral, which satisfies (15).

$$t_p(x,y) = HgPr\sum_{n=1}^{\infty} f_n(x)\cos\lambda_n y \tag{16}$$

where