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New formulations of the temperature defect law for turbulent boundary layers on a plate

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ABSTRACT

A consistent asymptotic theory describing hydrodynamic and thermal turbulent boundary layers on a flat plate in zero pressure gradient is developed. The fact that the flow depends on a limited number of governing parameters allows us to formulate algebraic closure conditions that relate the turbulent shear stress and turbulent heat flux to mean velocity and temperature gradients. As a result of an exact asymptotic solution of the boundary-layer equations, the known laws of the wall for the velocity and temperature and the velocity and temperature defect laws as well as the expressions for the skin-friction coefficient, Stanton number, and Reynolds-analogy factor are obtained. The latter implies two new formulations for the temperature defect law one of which is completely similar to the velocity defect law and does not contain the Stanton number and the turbulent Prandtl number, and the other does not contain the skin-friction coefficient. A heat-transfer law is obtained that relates only thermal quantities. The theoretical conclusions agree well with experimental data.

1. Introduction

The well-known temperature defect law, valid in the outer region of the turbulent boundary layer, see e.g. [1], uses as a characteristic scale the so-called friction temperature, which is calculated from the skin-friction coefficient and Stanton number for which measurements of both wall shear stress and heat flux are needed. The same applies to the universal heat-transfer law [1], which relates three quantities: the Stanton number, the skin-friction coefficient, and a Reynolds number. For the turbulent boundary layer on a flat plate, such measurements, as far as we know, were performed in a single work [2].

In deriving the similarity laws [1], there are used only dimensional analysis and a single physical assumption [3], according to which the flow under consideration has two characteristic length scales: the viscous one (that determines the thickness of the viscous near-wall sublayer) and the outer one (boundary-layer thickness). At high Reynolds numbers, molecular viscosity and heat conductivity are not essential outside the viscous sublayer (and are not among the governing parameters for the flow outside the viscous sublayer) while the outer scale exerts no influence on the processes near the wall and is not a governing parameter in this region.

The present paper suggests a different approach to the classical problem under consideration, which is based on solving the momentum and heat-transfer equations, closure conditions for which are formulated (under the same physical assumption [3]) in terms of functional dependencies of the turbulent shear stress and turbulent heat flux upon velocity and temperature gradients. The existence of these functional relations is a consequence of the fact that the considered turbulent flow in whole depends only on a limited number of governing parameters. The idea of such a closure method was first formulated in Ref. [4] and then used in subsequent works, see e.g. Refs. [5–8].

Another essential element of the investigation is a special change of variables in the boundary-layer equations [9], which allows us to seek the solution to the problem in the form of asymptotic expansions in high values of the logarithm of the Reynolds number based on the boundary-layer thickness. As a result, along with the known similarity laws for the outer and wall regions of the boundary layer, the expression for the Reynolds-analogy factor is obtained which implies two new formulations of the temperature defect law. The first one is completely similar to the velocity defect law, i.e. contains neither the Stanton number nor the turbulent Prandtl number and the second one uses only thermal quantities, i.e. does not contain the skin-friction coefficient. A heat-transfer law is obtained that relates only thermal quantities. These relations can already be compared with a wider set of experimental data. Thus, we use the work [10], in which the temperature profiles and skin friction are measured but there is no data on heat flux, and the paper [11], which reports Stanton numbers but contains no values of skin-friction coefficients.

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Thus, the purpose of the present work is to obtain similarity laws resting upon first principles without invoking any special hypotheses and approximate turbulent models. As it is well known, there is a great deal of literature devoted to the latter ones, and among this literature one can mention some of the latest works [12–14].

2. Problem formulation and closure conditions

We consider the flow of incompressible fluid in a turbulent boundary layer on a flat, smooth plate. The free stream has a velocity u_e and a temperature T_e. The plate temperature T_w is constant. The origin of a Cartesian coordinate system is at the leading edge of the plate.

2.1. Governing equations

The flow is described by the Navier–Stokes, continuity, and heat-transfer equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u}, \quad \nabla\mathbf{u} = 0, \tag{1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\nabla T = \chi\nabla^2 T. \tag{2}$$

For low-speed, incompressible flow, the kinetic heating associated with the viscous dissipation of energy is negligibly small and is not taken into account in Eq. (2).

The system of turbulent boundary-layer equations is obtained after averaging of Eqs. (1) and (2) and neglecting of a number of relatively small terms

$$\begin{aligned} &\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}\left(\nu\frac{\partial \bar{u}}{\partial y} - \overline{u'v'}\right), \\ &\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \frac{\partial}{\partial y}\left(\chi\frac{\partial \bar{T}}{\partial y} - \overline{T'v'}\right), \\ &\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0. \end{aligned} \tag{3}$$

2.2. Closure conditions

For incompressible flow, the hydrodynamic problem (1) is independent of the thermal one (2) and all hydrodynamic quantities, among them mean-velocity gradient and turbulent shear stress, are functions of the Cartesian coordinates x and y and three governing parameters—the density ρ, the kinematic viscosity ν, and the free-stream velocity u_e:

$$\frac{\partial \bar{u}}{\partial y} = F_1(x, y, \rho, \nu, u_e), \tag{4}$$

$$\overline{u'v'} = F_2(x, y, \rho, \nu, u_e). \tag{5}$$

Let us consider a boundary-layer thickness

$$\Delta = F_3(x, \rho, \nu, u_e) \tag{6}$$

as a quantity characterizing the transverse length scale of the turbulent flow. For example, as it is usually accepted in practice, we may consider the boundary-layer thickness as the distance from the wall δ_{99} at which the streamwise mean-velocity component differs from u_e by 1%. Solving Eqs. (4) and (6) for quantities x and u_e and substituting them into Eq. (5), we obtain

$$\overline{u'v'} = F_4\left(\Delta, y, \rho, \nu, \frac{\partial \bar{u}}{\partial y}\right).$$

After applying to this relation the Π-theorem, we have

$$\overline{u'v'} = -\left(y\frac{\partial \bar{u}}{\partial y}\right)^2 S(\mathrm{Re}, \eta), \quad \mathrm{Re} = \frac{y^2}{\nu}\frac{\partial \bar{u}}{\partial y}, \quad \eta = \frac{y}{\Delta}. \tag{7}$$

Here, S is a universal function and Re is the local Reynolds number based on the mean-velocity gradient and the distance from the wall.

Temperature field is calculated on the basis of velocity field, therefore governing parameters for the thermal problem are ρ, ν, u_e, the temperature-conductivity coefficient χ, and the temperature difference $T_w - T_e$:

$$\frac{\partial \bar{T}}{\partial y} = F_5(x, y, \rho, \nu, \chi, u_e, T_w - T_e), \tag{8}$$

$$\overline{T'v'} = F_6(x, y, \rho, \nu, \chi, u_e, T_w - T_e). \tag{9}$$

The transverse temperature gradient and turbulent temperature flux depend on the difference of the temperatures on the wall and in the free stream due to linearity of Eq. (2). Solving Eqs. (4), (6), and (8) for the quantities x, u_e, and $T_w - T_e$ and substituting them into Eq. (9), we obtain

$$\overline{T'v'} = F_7\left(\Delta, y, \rho, \nu, \chi, \frac{\partial \bar{u}}{\partial y}, \frac{\partial \bar{T}}{\partial y}\right). \tag{10}$$

Since, as mentioned before, we neglect the kinetic heating, the equation for temperature (2) is homogeneous. Therefore, one may choose for temperature an arbitrary dimension. Taking this into account, after applying the Π-theorem, we get

$$\overline{T'v'} = -y^2\frac{\partial \bar{T}}{\partial y}\frac{\partial \bar{u}}{\partial y}H(\mathrm{Re}, \mathrm{Pe}, \eta), \quad \mathrm{Pe} = \frac{y^2}{\chi}\frac{\partial \bar{u}}{\partial y}. \tag{11}$$

Here, H is a universal function and Pe is the local Péclet number based on the mean-velocity gradient and the distance from the wall. The Reynolds and Péclet numbers are related by the equation $\mathrm{Pe} = \mathrm{Pr}\mathrm{Re}$, where $\mathrm{Pr} = \nu/\chi$ is the molecular Prandtl number.

Two characteristic flow regions—the viscous sublayer and the outer region of the boundary layer—are described in terms of the variables Re, Pe, and η as follows. In the viscous sublayer, $\mathrm{Re} = O(1)$ and $\mathrm{Pe} = O(1)$ while the distance from the wall normalized with the boundary-layer thickness $\eta \to 0$. In the outer region of the boundary layer, $1/\eta = O(1)$ while the local Reynolds and Péclet numbers Re and Pe tends to infinity as $\nu \to 0$ and $\chi \to 0$. The functions S and H can be considered as continuous and differentiable functions of their arguments. We impose weaker conditions

$$\begin{aligned} &S(\mathrm{Re}, \eta) = S(\mathrm{Re}, 0) + O(\eta^{\alpha_1}), \quad \eta \to 0, \quad \mathrm{Re} = O(1), \\ &S(\mathrm{Re}, 0) = S(\infty, 0) + O(\mathrm{Re}^{-\alpha_2}), \quad \mathrm{Re} \to \infty, \\ &S(\mathrm{Re}, \eta) = S(\infty, \eta) + O(\mathrm{Re}^{-\alpha_2}), \quad \mathrm{Re} \to \infty, \quad 1/\eta = O(1), \\ &S(\infty, \eta) = S(\infty, 0) + O(\eta^{\alpha_1}), \quad \eta \to 0, \quad \alpha_1, \alpha_2 > 0, \end{aligned} \tag{12}$$

and similar conditions for the function $H(\mathrm{Re}, \mathrm{PrRe}, \eta)$. The conditions (12) are a mathematical formulation of the main physical assumption [3,1], according to which the flow under consideration has only two characteristic length scales: viscous one (which determines the viscous-sublayer thickness) and outer one (the boundary-layer thickness). At high Reynolds numbers, molecular viscosity and heat-conductivity are not essential outside the viscous sublayer while the outer scale exerts no influence on flow in the wall region.

3. Wall region

Integration of the momentum and energy Eq. (3) across the layer in view of the continuity equation and closure conditions (7) and (11) yields