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Highly efficient and local projection-based stabilized finite element method for natural convection problem ☆

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ABSTRACT

Based on two local Gauss integrations and the projection-based stabilized finite element method, a local projection-based stabilized finite element method for natural convection problem is proposed. The Oseen iteration and decoupling technique are applied to improve stability and save computational time. Compared with the common projection-based stabilized finite element method, the new method does not need to introduce any extra degree of freedom. Thus, it can save a large amount of CPU-time to get the same precision. Numerical results on the known analytical solutions, the driven cavity flow and the partitioned square enclosure problem are given to verify the theoretical predictions and demonstrate its high efficiency.

1. Introduction

The natural convection problem constitutes an important system of equations in atmospheric dynamics atmospheric fronts, katabatic winds, natural ventilation, solar collectors, dense gas dispersion, cooling of electronic equipments and nuclear reactors, and insulation with double pane window. This system does not only incorporate the velocity vector field as well as the pressure field but also contains the temperature field. Thus, development of an efficient and effective computational method for investigating this problem has practical significance, and has drawn the attention of many researchers.

At the time of writing, there are numerous works devoted to the development of efficient methods for the natural convection problem ([1–11] and the references therein). In the meantime, some numerical analysis and numerical results for the natural convection equations can be found in literature [1,2] by Boland and Layton. An explicit finite element algorithm for convection heat transfer problems has been presented by Manzari [12]. He has used a standard Galerkin finite element method for spatial discretization and an explicit multistage Runge–Kutta scheme to march in the time domain. In addition, Çıbık and Kaya [3] have formulated a projection-based stabilization finite element technique for solving the steady-state natural convection problems. The global stabilizations are added for both velocity and temperature variables and these effects are subtracted from the large scales.

This projection-based stabilization method, used projections into appropriate function spaces in order to decompose solution scales, can be considered as a subgrid stabilization method. The subgrid stabilization methods are generally based on the notion of scale separation which assumes that there exist large scales and small scales of the flow. The key ideas of the subgrid stabilization methods are to first split the approximation space into resolved scales and subgrid scales, and then slightly modify the Galerkin approximation by adding an asymptotically consistent artificial diffusion term on the subgrid scales [13,14,16–19]. The variational multiscale methods are also based on the decomposition of the flow scales and define the large scales by projection into appropriate subspaces; see Hughes et al. [20], John et al. [21], Kaya and Rivière [22], and Zheng et al. [23–25]. It is noted that due to the same underlying concept, many subgrid stabilization methods can be regarded as variational multiscale methods.

Standard Galerkin finite element method for natural convection problem usually yields inaccurate approximate solutions and may exhibit global spurious oscillations [3]. Hence, Çıbık and Kaya [3] have constructed the efficient projection-based stabilized finite element method for the natural convection problem to avoid some drawbacks of the classical methods. However, this method needs

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to add extra storage compared with standard Galerkin finite element method which introduces additional dependent variables. Thus, this paper aims to propose a local projection-based stabilization method for the natural convection equations without adding any degree of freedom. This novel method focuses on the projection-based stabilized finite element method based on two local Gauss integrations. The local Gauss integrations technique [26,27] was used as a stabilizing term for the Stokes and Navier–Stokes problems, where the authors applied them to offset the discrete pressure space by the residual of the simple and symmetry term at element level in order to circumvent the inf-sup condition. Furthermore, in [23,28], this technique was used to solve the incompressible flows problem at high Reynolds numbers and the viscoelastic flows.

The focus of this paper is to apply a decoupling technique to the natural convection problem. First, we propose a local projection-based stabilization method by replacing the auxiliary terms with two local Gauss integrations of the velocity and temperature in the classical projection-based stabilization formulations, and show that these two methods are equivalent in mathematics. However, the proposed algorithm does not need to introduce any extra degree of freedom, and can save a large amount of computational time. Further, to save much more CPU-time, a decoupling technique is used based on the local projection-based stabilization method.

This work can be cast in the framework of Zheng et al. [23] and Wang [28]. However, it is different from them because of the different and more complicated equations and the two local Gauss integrations technique for two variables (the velocity and temperature), not one variable (the velocity). In fact, this paper can be considered as a sequel and a complement of the work in [23,28]. The article is organized as follows. In next section, we introduce considered equations, notations and some well-known results used throughout this paper. Subsequently, a local projection-based stabilized finite element method based on two local Gauss integrations is introduced in Section 3. Moreover, numerical examples are shown to verify the good properties of our method in Section 4. Finally, we end with a short conclusion.

2. Preliminaries

Let Ω be a bounded open domain in $\mathbb{R}^2$ with disjoint domains Ω_s and Ω_f, assumed to have a Lipschitz continuous boundary $\partial\Omega$. Suppose $\Gamma_T = \partial\Omega \setminus \Gamma_B$, where Γ_B is a regular open subset of $\partial\Omega$. Consider the following stationary natural convection equations including solid media in dimensionless form [1–3]

$$\begin{aligned}
&-Pr\Delta u + (u\cdot\nabla)u + \nabla p = Pr\,Ra\,jT \quad \text{in } \Omega_f,\\
&\operatorname{div} u = 0 \quad \text{in } \Omega_f,\\
&-\nabla\cdot(\kappa\nabla T) + (u\cdot\nabla)T = \gamma \quad \text{in } \Omega,\\
&u = 0 \quad \text{on } \partial\Omega_f, \quad u \equiv 0 \quad \text{in } \Omega - \Omega_f = \Omega_s,\\
&T = 0 \quad \text{on } \Gamma_T, \quad \frac{\partial T}{\partial n} = 0 \quad \text{on } \Gamma_B,
\end{aligned} \tag{1}$$

where $u = (u_1(x), u_2(x))$ represents the velocity vector, $p = p(x)$ the pressure, $T(x)$ the temperature, γ the forcing function, $Pr, Ra > 0$ the Prandtl and Rayleigh numbers, $\kappa > 0$ the thermal conductivity parameter, $j = (0,1)$ the two-dimensional vector, and n the outward unit normal to the Γ_B. Besides, the symbols Δ, ∇ and div denote the Laplacian, gradient and divergence operators, respectively. Moreover, take the case $\kappa \equiv \kappa_f$ in Ω_f and $\kappa \equiv \kappa_s$ in Ω_s, where κ_f and κ_s are positive constants denoted the thermal conductivity for the different media.

For the mathematical setting of problem (1), we introduce the following Hilbert spaces:

$$X = H_0^1(\Omega_f)^2, \quad W = \{s \in H^1(\Omega) : s = 0 \quad \text{on } \Gamma_B\},$$
$$M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q\,\mathrm{d}x = 0 \right\}.$$

The space $L^2(\Omega)$ is equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. The space X is endowed with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\|\nabla u\|_0$. Standard definitions are used for the Sobolev spaces $W^{m,p}(\Omega)$, with the norm $\|\cdot\|_{m,p}, m, p \geqslant 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$.

We define two continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$, respectively, by

$$a(u,v) = (\nabla u, \nabla v), \quad \forall u, v \in X, \qquad d(v,q) = (q, \operatorname{div} v), \quad \forall v \in X, \ \forall q \in M,$$

and a trilinear form on $X \times X \times X$ by

$$\begin{aligned}
b(u;v,w) &= ((u\cdot\nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w)\\
&= \frac{1}{2}b_1(u;v,w) - \frac{1}{2}b_1(u;w,v), \quad \forall u, v, w \in X,
\end{aligned}$$

where $b_1(u;v,w) = ((u\cdot\nabla)v, w)$. For fixed u, note that $b(u;v,w)$ is the skew-symmetric part of $b_1(u;v,w)$.

We also define a continuous bilinear form $\bar{a}(\cdot,\cdot)$ and a trilinear form $\bar{b}(\cdot;\cdot,\cdot)$ on $W \times W$ and $X \times W \times W$, respectively, by

$$\bar{a}(T,s) = (\nabla T, \nabla s), \quad \forall T, s \in W,$$

and

$$\begin{aligned}
\bar{b}(u;T,s) &= ((u\cdot\nabla)T, s) + \frac{1}{2}((\operatorname{div} u)T, s)\\
&= \frac{1}{2}\bar{b}_1(u;T,s) - \frac{1}{2}\bar{b}_1(u;s,T), \quad \forall u \in X, \quad \forall T, s \in W,
\end{aligned}$$

where $\bar{b}_1(u;T,s) = ((u\cdot\nabla)T, s)$.

With the above notations, the variational formulation of problem (1) reads as follows: Find $(u,p,T) \in X \times M \times W$ such that for all $(v,q,s) \in X \times M \times W$,

$$\begin{cases}
Pr\,a(u,v) - d(v,p) + d(u,q) + b(u;u,v) = Pr\,Ra\,(jT, v),\\
\kappa\bar{a}(T,s) + \bar{b}(u;T,s) = (\gamma, s).
\end{cases} \tag{2}$$

3. A local projection-based stabilized finite element method

Let $\mathcal{T}_h = \{K\}$ be a regular triangulation of Ω, indexed by a parameter $h = \max_{K\in\mathcal{T}_h}\{\operatorname{diam}(K)\}$. We consider the finite element spaces

$$X_h = \left\{ v_h \in X \cap C^0(\bar\Omega)^2 : v_h|_K \in P_2(K)^2, \quad \forall K \in \mathcal{T}_h \right\},$$
$$M_h = \left\{ q_h \in M \cap C^0(\bar\Omega) : q_h|_K \in P_1(K), \quad \forall K \in \mathcal{T}_h \right\},$$
$$W_h = \left\{ s_h \in W \cap C^0(\bar\Omega) : s_h|_K \in P_2(K), \quad \forall K \in \mathcal{T}_h \right\},$$
$$Y_h = \left\{ r_h \in L^2(\Omega) : r_h|_K \in P_0(K), \quad \forall K \in \mathcal{T}_h \right\},$$

where $P_i(K)$ represents the space of ith order polynomial on the set $\mathcal{T}_h, i = 0,1,2$. Set $L_h = Y_h(\Omega)^{2\times 2}$, and $R_h = Y_h(\Omega)^{1\times 2}$. Note that $X_h \times M_h$ satisfies the following discrete inf-sup condition

$$\sup_{v\in X_h} \frac{d(v,q)}{\|\nabla v\|_0} \geqslant \beta\|q\|_0, \quad \forall q \in M_h,$$

where the constant $\beta > 0$ is independent of h.

We firstly consider a common version of projection-based stabilized finite element method which was proposed in [3] as follows: