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A precise integration boundary element method for solving transient heat conduction problems

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ABSTRACT

In this paper, a combined approach of boundary element method and precise integration method is presented for solving transient heat conduction problems with heat sources. The boundary integral equation is derived by means of the Green's function for the Laplace equation, and as a result, two domain integrals are involved in the derived integral equations. Firstly, the radial integration method is used to convert the domain integrals into equivalent boundary integrals, so the system of ordinary differential equations on the boundary integral equation can be obtained by the boundary element method. Then, the precise integration method is adopted to solve the system of ordinary differential equations. Finally, several numerical examples are given to demonstrate the performance of the present method. The results show that the present approach gives satisfactory performance even for very large time step size, and the results given by the present approach are independent of the time step size if some integrals involves boundary conditions and heat sources can be integrated analytically.

1. Introduction

The transient heat conduction problems occur widely in engineering practice. Generally, only linear problems with simple geometries and boundary conditions can lead to analytic results [1]. Therefore, in the past half century, many works have been done to develop effective numerical methods for solving these problems.

The transient heat conduction problems include the variables of space and time. Currently, the space discretized methods include mainly the finite difference method (FDM) [2], the finite element method (FEM) [3], the meshless method [4], the finite volume method (FVM) [5] and the boundary element method (BEM) [6,7]. For discretizing the time domain, in most cases the finite difference is used to replace the derivative of time [8]. It is well known that the FDM is very general to solve the time-dependent problems. However, the accuracy of results is sensitive to the time step size in FDM [2]. In addition, for the general finite-difference time-domain method [9], Courant–Friedrich–Levy (CFL) condition [10] must be satisfied when a time step size is used.

In 1994, Zhong and Williams [11] proposed the precise integration method (PIM). Not only the method can obtain the stable and high precision results, but also the results are independent of the time step size when the free term can be divided into the functions of space and time. It is worthy to emphasize that the method is easy to implement. Up to now, the method combining the PIM with the FEM has been applied to many fields, such as the effect of nonlinear contact upon natural frequency of composite plate [12], the transient forced vibration analysis of beams [13] and the sensitivity analysis and optimization problem [14]. In addition, the method combining the PIM with meshless local Petrov–Galerkin method has been applied to the transient heat conduction problems [15].

Compared with FDM, FEM, FVM and the meshless method, BEM is very robust for solving the linear and homogeneous heat conduction problem [16–18]. However, solving nonlinear problems is still a challenge task by BEM, since the fundamental solutions of these problems can be obtained only for some very special cases [19–22]. Fortunately, we can use the fundamental solution of the linear problem to solve the nonlinear problem, whereas domain integrals are involved in resulting integral equations.

To overcome this difficulty, the dual reciprocity method (DRM) [23] and multiple reciprocity method (MRM) [24] have been used to transform the domain integrals into the boundary integrals. In this method, the transformation is carried out by approximating the source term with a series of basis functions and using their particular solutions. The DRM has been extensively used to solve the nonlinear and nonhomogeneous problems [25]. The deficiency of this technique is that the particular solutions may be difficult to

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obtain for some complicated problems. In addition, even for known heat sources term, the method still requires an approximation of the known function [25].

In 2002, Gao [26,27] presented a very robust new transformation technique, which called the radial integration method (RIM). The RIM not only can transform any complicated domain integral to the boundary without using particular solution, but also can remove various singularities appearing in domain integrals [28]. The main feature of RIM is that it can treat different types of domain integrals appearing in the same integral equation in a unified way, since it does not resort to particular solutions as in the DRM. The method combining the RIM with the BEM is called the radial integration boundary element method (RIBEM) [29,30].

The RIBEM has been widely applied to many fields including the dynamic analysis of laminate composite plates [31], the nonlinear and nonhomogeneous elastic problems [32], the crack analysis in functionally graded materials [33], the viscous flow problems [34], the one-phase solidification problem [35] and the heat conduction problems [29,30,36]. However, when solving time-dependent problems via RIBEM, the results are sensitive for different time step size due to using the finite difference technique to replace the derivative term with respect to time.

In this paper, the PIM and the RIBEM (it will be abbreviated as PIBEM) are combined to solve transient heat conduction problems with heat sources. First of all, we discretize space domain by using the RIBEM to obtain a system of ordinary differential equations (ODEs) with respect to time, and then solve the ODEs by the PIM. Finally, three examples are presented to validate the proposed method.

2. Governing equations

Considering a two-dimensional bounded domain Ω with constant material parameters, the governing equation for transient heat conduction problems in isotropic media can be expressed as

$$k\nabla^2 T(\mathbf{x},t) + f(\mathbf{x},t) = \rho c \frac{\partial T(\mathbf{x},t)}{\partial t}, \quad \mathbf{x} \in \Omega \tag{1}$$

where $\mathbf{x} = (x_1, x_2), \nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplace operator, $T(\mathbf{x},t)$ is the temperature at point $\mathbf{x} \in \Omega$ and at time t, k is the thermal conductivity, $f(\mathbf{x},t)$ is a known heat source at time t, ρ is the density and c is the specific heat.

The initial condition is

$$T(\mathbf{x},0) = T_0 \tag{2}$$

where T_0 is a prescribed function. The boundary conditions are

$$T(\mathbf{x},t) = \overline{T}(\mathbf{x},t) \quad \mathbf{x} \in \Gamma_1 \tag{3}$$

$$-k\frac{\partial T}{\partial n} = \bar{q}(\mathbf{x},t) \quad \mathbf{x} \in \Gamma_2 \tag{4}$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma, \Gamma_1 \cap \Gamma_2 = \varnothing, \Gamma = \partial\Omega, \overline{T}$ and $\bar{q}$ are prescribed temperature history and flux on the boundary, respectively.

3. Numerical implementation of the RIBEM

3.1. Boundary-domain integral equation

To derive the boundary integral equation, a weight function G is introduced to Eq. (1) and the following domain integrals can be written as

$$\int_\Omega Gk\nabla^2 T d\Omega + \int_\Omega Gf d\Omega = \int_\Omega G\rho c \frac{\partial T}{\partial t} d\Omega \tag{5}$$

Using Gauss' divergence theorem, the first domain integral can be manipulated as

$$\int_\Omega Gk\nabla^2 T d\Omega = k\int_\Gamma \left(G\frac{\partial T}{\partial n} - T\frac{\partial G}{\partial n}\right) d\Gamma + k\int_\Omega T\nabla^2 G d\Omega \tag{6}$$

If the weight function G is Green's function $G(\mathbf{x},\mathbf{y})$ which satisfies the following equation:

$$\nabla^2 G + \delta(\mathbf{x}-\mathbf{y}) = 0 \tag{7}$$

where $\delta(\mathbf{x}-\mathbf{y})$ is the Dirac delta function, according to literature [37], the Green's function $G(\mathbf{x},\mathbf{y})$ in Eq. (7) can be expressed as

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\ln\frac{1}{r(\mathbf{x},\mathbf{y})} \tag{8}$$

where $r(\mathbf{x},\mathbf{y})$ is the distance between the source point $\mathbf{y}$ and the field point $\mathbf{x}$.

Then in terms of the integration property of the Dirac delta function, the domain integral of Eq. (6) can be written as

$$\int_\Omega T\nabla^2 G d\Omega = -T(\mathbf{y}) \tag{9}$$

Substituting this equation into Eqs. (6) and (5), it follows that

$$kT(\mathbf{y}) = k\int_\Gamma \left(G\frac{\partial T}{\partial n} - T\frac{\partial G}{\partial n}\right) d\Gamma + \int_\Omega Gf d\Omega - \rho c\int_\Omega G\frac{\partial T}{\partial t} d\Omega \tag{10}$$

Boundary-domain integral Eq. (10) is valid only for internal points. For boundary points, a similar integral equation can be obtained by letting $\mathbf{y} \to \Gamma$ as is done in the conventional BEM [37]. A general integral equation is presented as following:

$$c(\mathbf{y})kT(\mathbf{y}) = k\int_\Gamma \left(G\frac{\partial T}{\partial n} - T\frac{\partial G}{\partial n}\right) d\Gamma + \int_\Omega Gf d\Omega - \rho c\int_\Omega G \times \frac{\partial T}{\partial t} d\Omega \tag{11}$$

where

$$c(\mathbf{y}) = \begin{cases} 1, & \mathbf{y} \in \Omega \\ \frac{\varphi(\mathbf{y})}{2\pi}, & \mathbf{y} \in \Gamma \end{cases} \tag{12}$$

$\varphi(\mathbf{y})$ is the interior angle at a point $\mathbf{y}$ of the boundary Γ. Particularly, $c(\mathbf{y}) = 0.5$ if $\mathbf{y}$ is a smooth point on the boundary.

3.2. Transformation of domain integrals to the boundary by RIM

The two domain integrals involved in Eq. (11) are transformed into equivalent boundary integrals by RIM [26–28]. In order to describe clearly the process, we assume that the boundary Γ is discretized into N_b linear elements, the domain Ω is distributed N_I internal nodes and the total number of nodes is $N = N_b + N_I$.

In general, the heat source $f(\mathbf{x},t)$ is a known function. Therefore, the RIM can be directly used to transform the first domain integral in Eq. (11) to the boundary as follows:

$$\int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{x},t)d\Omega(\mathbf{x}) = \int_\Gamma \frac{1}{r(\mathbf{z},\mathbf{y})}\frac{\partial r}{\partial n}F^A(\mathbf{z},\mathbf{y},t)d\Gamma(\mathbf{z}) \tag{13}$$

where the radial integral F^A can be expressed as

$$F^A(\mathbf{z},\mathbf{y},t) = \int_0^{r(\mathbf{z},\mathbf{y})} G(\mathbf{x},\mathbf{y})f(\mathbf{x},t)\xi d\xi \tag{14}$$

In Eqs. (13) and (14), it is noted that $\mathbf{z}$ is the boundary point and the variable transformation relationship about $\mathbf{x}$ can be given by [26]

$$\mathbf{x} = \mathbf{y} + \hat{\mathbf{r}}\xi \tag{15}$$

In Eq. (15), $\hat{\mathbf{r}} = (\mathbf{z}-\mathbf{y})/r(\mathbf{z},\mathbf{y})$ is a unit vector or it can also be expressed as a component form, i.e., $x_i = y_i + r_{,i}\xi$ where $r_{,i} = \partial r/\partial x_i = (z_i - y_i)/r(\mathbf{z},\mathbf{y})$ subscript $i = 1, 2$. The radial integral F^A can be evaluated analytically or numerically by using the above