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Review

Stability and Second Law of Thermodynamics in dual-phase-lag heat conduction ☆

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ABSTRACT

In this paper, two heat conductor models with fading memory are obtained by different approximations of Tzou's dual-phase-lag theory. For the first model, we obtain a close coincidence between the restrictions of the second law and the asymptotic stability. Otherwise, for the second model the thermodynamic restrictions appear more restrictive compared with the conditions following from the asymptotic stability.

Contents

1. Introduction

In the last years, there have been considered several constitutive equations for the heat flux, obtained by different approximations of the theory proposed by Tzou [20–22] in 1995. In which the classical Fourier law is replaced by the equation

$$q(x,t+\tau_q) = -k\nabla\theta(x,t+\tau_\theta) \quad \tau_q > 0 \quad \tau_\theta > 0. \tag{1}$$

Even recently, it has been shown that the related differential problem can lead to ill posed problems (see [4]). Otherwise, it was proved that the model (1) for $\tau_q \neq \tau_\theta$ is not in agreement with the Second Law of Thermodynamics (see [10]).

Nevertheless, some differential problems obtained by suitable expansions of Eq. (1) can satisfy the thermodynamic restrictions. These models are obtained by taking into account the Taylor series in both sides of (1) an retaining terms up a fixed order in τ_q and τ_θ.

In this paper we consider the following approximate constitutive equations for the heat flux

$$\frac{\tau_q^2}{2}\ddot{q}(x,t) + \tau_q\dot{q}(x,t) + q(x,t) = -k[\nabla\theta(x,t) + \tau_\theta\nabla\dot{\theta}(x,t)], \tag{2}$$

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$$\frac{\tau_q^2}{2}\ddot{q}(x,t)+\tau_q\dot{q}(x,t)+q(x,t)=-k\left[\nabla\theta(x,t)+\tau_\theta\nabla\dot{\theta}(x,t)+\frac{\tau_\theta^2}{2}\nabla\ddot{\theta}(x,t)\right]. \tag{3}$$

The heat equation, associated with the constitutive equation (2) or (3), determines partial differential equations of the third order in time which have received attention in the literature for the study of existence, uniqueness and stability, under suitable restrictions on the parameters τ_q and τ_θ (see for example Quintanilla [15,16] and Wang et al. [24–26]).

The aim of this paper is to rewrite Eqs. (2) and (3) in the framework of Gurtin–Pipkin and Coleman–Gurtin fading memory theory [13,6]. So that, the heat flux depends on the history of the temperature gradient.

With this different approach we the study the compatibility of the models with thermodynamical principles. We observe a natural agreement between the conditions on the asymptotic stability and the thermodynamic constrains for the model (2), indeed we obtain the same restrictions on τ_q and τ_θ.

Otherwise, in the study of Eq. (3), written by a fading memory constitutive equation, we prove the exponential stability under the same conditions $\tau_q<(2+\sqrt{3})\tau_\theta$ considered by Quintanilla [15], while the restrictions which follow from Second Law involve the following more restrictive inequality

$$\left(2-\sqrt{3}\right)\tau_\theta<\tau_q<\left(2+\sqrt{3}\right)\tau_\theta. \tag{4}$$

Otherwise, from the second example, we observe as the stability conditions appear wider than the thermodynamic restrictions (4), which follow from the Second Law expressed on closed cycles.

In the past, the relationship between asymptotic stability and thermodynamic restrictions has attracted the interest of many researches with very meaningfull papers [9,5,12,7,8]. In these works, it was observed a harmony between stability and thermodynamic requirements. Therefore a first glance, the results of this paper appear unusual, because they conflict with these previous important researches. For this reason the example studied above can be of some interest and has to require further study, in order to verify if the thermodynamic conditions may be affected by the particular expression used for the Second Law.

2. Dual-phase-lag heat conductor model (2) as a material with memory

It is well known that the Cattaneo–Maxwell constitutive equation [3]

$$\tau_q\dot{q}(x,t)+q(x,t)=-k\nabla\theta(x,t),$$

can be rewritten in the context of the Gurtin–Pipkin theory in the following form

$$q(x,t)=-\int_{-\infty}^{t}\kappa(t-s)\nabla\theta(x,s)\,ds, \tag{5}$$

where the memory kernel κ is given by

$$\kappa(s)=\frac{k}{\tau_q}e^{-\frac{s}{\tau_q}}.$$

Later on the dependence on x will be omitted.

We observe that $\xi(t)=ae^{-t/\tau_q}$ is a general solution of equation

$$\tau_q\dot{\xi}(t)+\xi(t)=0.$$

To rewrite Eq. (2) as a memory constitutive model, we observe that the general solution of equation

$$\frac{1}{2}\tau_q^2\ddot{\xi}(t)+\tau_q\dot{\xi}(t)+\xi(t)=0,$$

is given by

$$\xi(t)=e^{-t/\tau_q}[a\cos(t/\tau_q)+b\sin(t/\tau_q)].$$

So that, when we put

$$q(t)=-\int_{-\infty}^{t}e^{-(t-s)/\tau_q}\left[a_1\cos\left(\frac{t-s}{\tau_q}\right)+b_1\sin\left(\frac{t-s}{\tau_q}\right)\right]\nabla\theta(s)\,ds, \tag{6}$$

then the constants a_1 and b_1 satisfy (2) if the equation holds

$$\frac{1}{2}\tau_q^2\ddot{q}(t)+\tau_q\dot{q}(t)+q(t)=-\left[\frac{a_1+b_1}{2}\tau_q\nabla\theta(t)+\frac{a_1}{2}\tau_q^2\nabla\dot{\theta}(t)\right]. \tag{7}$$

The comparison of (7) and (2) yields

$$a_1=2k\frac{\tau_\theta}{\tau_q^2},\quad b_1=2k\frac{\tau_q-\tau_\theta}{\tau_q^2}.$$

With these values, (6) become

$$q(t)=-\frac{2k}{\tau_q^2}\int_{-\infty}^{t}e^{-(t-s)/\tau_q}\left[\tau_\theta\cos\left(\frac{t-s}{\tau_q}\right)+(\tau_q-\tau_\theta)\sin\left(\frac{t-s}{\tau_q}\right)\right]\nabla\theta(x,s)\,ds,$$

or in the equivalent form

$$q(t)=-\frac{k}{\tau_q^2}\int_0^\infty\kappa_1(s)\nabla\theta^t(x,s)\,ds, \tag{8}$$

where $\theta^t(x,s)=\theta(x,t-s)$ and

$$\kappa_1(s)=2e^{-s/\tau_q}\left[\tau_\theta\cos\left(\frac{s}{\tau_q}\right)+(\tau_q-\tau_\theta)\sin\left(\frac{s}{\tau_q}\right)\right]. \tag{9}$$

2.1. Thermodynamic restrictions

To determine the restrictions imposed by thermodynamics on the constitutive Eq. (8), we postulate the Second Law of Thermodynamics in terms of Clausius–Duhem inequality [11].

In the framework of linear rigid conductor with memory, this inequality can be formulated on cyclic histories of period T by requiring (see, for example, [1])

$$\oint q(t)\cdot\nabla\theta(t)\,dt\leqslant 0, \tag{10}$$

where the equality occurs only for the null cycle.

Consequently, any cycle characterized by the history

$$\nabla\theta^t(s)=g_1\cos[\omega(t-s)]+g_2\sin[\omega(t-s)], \tag{11}$$

with $\omega>0$ and $|g_1|^2+|g_2|^2>0$ must satisfy (10) as an inequality.

When q is given by (8) and periodic histories (11), inequality (10) becomes

$$-\frac{k}{\tau_q^2}\int_0^{2\pi/\omega}[g_1\cos(\omega t)+g_2\sin(\omega t)]\cdot\int_0^\infty\kappa_1(s)\{g_1\cos[\omega(t-s)]+g_2\sin[\omega(t-s)]\}\,ds\,dt<0,$$

from which, we obtain

$$-\frac{k}{\tau_q^2}\frac{\pi}{\omega}(g_1^2+g_2^2)\int_0^\infty\kappa_1(s)\cos(\omega s)\,ds<0.$$

Since the half range cosine Fourier transform of κ_1 is

$$\kappa_{1_c}(\omega)=\frac{2\tau_q^2}{4+\omega^4\tau_q^4}\left[2+(2\tau_\theta-\tau_q)\tau_q\omega^2\right],$$

hence the thermodynamic restriction (10) is satisfies if