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Fractional Stefan problems

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ABSTRACT

The solution of the classical one-dimensional Stefan problem predicts that in time t the melt front goes as $s(t) \sim t^{\frac{1}{2}}$. In the presence of heterogeneity, however, anomalous behavior can be observed where the time exponent $n \neq \frac{1}{2}$. In such a case, it may be appropriate to write down the governing equations of the Stefan problem in terms of fractional order time ($1 \geqslant \beta > 0$) and space ($1 \geqslant \alpha > 0$) derivatives. Here, we present sharp and diffuse interface models of fractional Stefan problems and discuss available analytical solutions. We illustrate that in the fractional time case ($\beta < 1$), a solution of the diffuse interface model in the sharp interface limit will not coincide with the solution of the sharp interface counterpart; negating a well know result of integer derivative Stefan problems. The paper concludes with the development of an implicit time stepping numerical solution for the diffuse interface fractional Stefan model. Results from this solution are verified with available analytical solutions.

1. Introduction

The one-phase Stefan problem, involving the tracking of the phase change interface $s(t)$ during the melting of a solid initially at the phase change temperature $T = 0$, is the archetypical moving boundary problem [1,2]. The diffusion like heat flux in this problem can be expressed in terms of a local instantaneous temperature gradient. As such, based on the analogy between diffusion and Brownian motion, the time scaling of the problem—which will determine the nature of the movement of the phase front—is $\sim t^{\frac{1}{2}}$. In the presence of media heterogeneity, however, fast transport paths and/or regions of hold-up can result in anomalous diffusion where the exponent in the time scale $n \neq \frac{1}{2}$. When the length scales of the heterogeneity are distributed as a power-law, Metzler and Klafter [3] have shown, by considering non-Brownian random walk processes, that anomalous diffusion can be modeled in terms of fractional derivatives [4]. A treatment that essentially represents the flux at a point in space and time as a non-local quantity made up of a weighted average of temperature gradients over space and through time; the former leading to super-diffusion behavior where $n > \frac{1}{2}$, the later to sub-diffusion behavior where $n < \frac{1}{2}$. Anomalous behaviors are observed in Stefan-like problems. In the horizontal diffusion of moisture in to a porous brick—a particular limit case of the Stefan problem see [5]—experiments measuring the movement of the moisture front $s(t) \sim t^n$ have observed both sub-($n < \frac{1}{2}$) and super-($n > \frac{1}{2}$) diffusion, see Table 1 in Sun et al. [6]. Experiments have also shown that the growth of the frost on a cooled plate can be super-diffusive, e.g., Tao et al. [7], arrive at an empirical correlation that gives the thickness of the frost layer growing as $\sim t^{0.655}$. Thus, there is some physical motivation to study anomalous Stefan problems. Further, when one notes that the distribution and connectivity of the pores in brick and the variations in the crystal morphology in a frost layer will have a distribution of length scales, it is reasonable to try and construct appropriate models using fractional derivatives.

Fractional Stefan and related moving boundary problems have been previously studied [8–10]. Liu and Xu [8] provide a closed form solution for the one-dimensional problem where the first order time derivative in the governing Fourier heat transfer equation and the Stefan moving boundary condition are replaced by fractional derivatives of order $1 \geqslant \beta > 0$. The result is a sub-diffusive movement of the phase front $s(t) \sim t^{\frac{\beta}{2}}$. In addition to using fractional time derivatives, Li et al. [9] also replace the Laplacian in the Fourier equation with an operator of fractional order $2 \geqslant 1 + \alpha > 1$ and model the flux term in the Stefan condition with a space derivative of order α. In this case, the front movement is given by $s(t) \sim t^{\frac{\beta}{1+\alpha}}$, which—depending on the choice of α and β—can exhibit both super and sub diffusive behaviors. Voller analytically studies the limit case one-dimensional Stefan problem related to the vertical infiltration of a sharp moisture front into soil [11] (the Green Ampt problem) and the horizontal movement of moisture in a porous brick [5]. The later of these works replacing the first order time derivative with a fractional derivative of order $1 \geqslant \beta > 0$ and modeling the flux as a fractional space derivative (fractional gradient) of order $1 \geqslant \alpha > 0$. Similar to the result in Li

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Nomenclature

a	constant	X	normalized space $[x/s]$
C	specific heat [J/kg K]	x	space
C_1	constant	α	order of space derivative $(0,1]$
c	Stefan number $[CT/L]$	β	order of time derivative $(0,1]$
f	liquid fraction	$\Delta x, \Delta t$	space and time steps
g	step function	ϵ	interface thickness
H	enthalpy	ρ	density [kg/m^3]
K	conductivity [J/m s K]		
k	weight	*subscripts/superscripts*	
L	latent heat [J/kg]	i, j	indices
ℓ	length scale	*it*	iteration counter
p	front parameter	*ipos*	nodal location of front
s	position of melt front	*jtim*	time step
T	temperature	n	time exponent
T_0	temperature at $x = 0$ [K]	*old*	old time step value
t	time	$*$	dimensioned quantity

et al. [9], the solution of this limit case also exhibits a time exponent with the super/sub diffusive form $\frac{\beta}{1+\alpha}$.

In terms of approximate solutions of fractional Stefan problems Li et al. [12] study a similarity form in a cylindrical geometry, Singh et al. [13] and Rajeev and Kushwaha [14] investigate the use of homotopy perturbation methods. The former of these last two works looks at the case where both time and space derivatives are fractional but offers no verification of the solutions. The later work just looks at the case of a fractional time derivative, the resulting approximate solution having the correct time exponent $n = \frac{\beta}{2}$ but failing to match the exact solution reported in [8].

The work noted above is based on the "sharp" interface form of the Stefan problem, where a specific condition, the Stefan condition, is used to determine the movement of the melt front. The alternative model is to use a "diffuse" interface model [1]. Here the sharp interface is smeared out by assuming that there is a temperature region $\epsilon \geqslant T \geqslant 0$ over which the liquid fraction—a known (prescribed) function of temperature $f(T)$—changes smoothly from the liquid value of $f = 1$ to the solid value of $f = 0$. In this way, the time evolution of the phase change can be tracked by considering the transient of the appropriate conserved quantity. In heat transfer settings this conserved quantity is the enthalpy, which, on suitable scaling, can be written as $H = cT + f(T)$ where c is a dimensionless specific heat term (i.e. the Stefan number—a ratio of sensible to latent heat). Physical realizations occur in the solidification of alloys and, in the appropriate limit, the transport of water in the partially saturated vadose zone. A number of studies of fractional forms of the diffusive interface models have been presented in the literature. For example, on defining the heat flux as a fractional gradient, Voller [15] and Damor et al. [16] use an explicit time stepping numerical treatment of the enthalpy form to model the solidification in a binary alloy; the work in [15] indicating the expected time scale $\sim t^{\frac{1}{1+\alpha}}$. Further, it can be shown that on interpreting the temperature as a metric potential and using a suitable form of the function $f(t)$ the limit case of the enthalpy formulation as $c \to 0$ matches the Richards equation for flow in partially saturated media. Time fractional versions of the Richards equation have been studied by Gerolymatou et al. [17] and in a related approach, based on fractal derivatives, by Sun et al. [6]; both studies indicating the expected time scale $\sim t^{\frac{\beta}{2}}$.

In a local and instantaneous setting it is known that as the thickness of the interface is reduced, i.e., as $\epsilon \to 0$, the solution of the diffusive interface model converges to the solution of the corresponding sharp interface model [1]. Recent work by Voller et al. [18] however, suggests that when a fractional time derivative is introduced into the enthalpy formulation, while the correct time scaling exponent is preserved, the solutions of the diffuse interface model in the limit $\epsilon \to 0$ and the sharp interface model are not in agreement. A situation that comes about through the different natures by which memory of the phase transition is accounted for. In the time fractional sharp interface model the memory of the phase transition is held by the moving interface, while in the time fractional diffuse model the memory is distributed throughout the melt region [18].

Although, for the sharp interface case, some approximate solutions of fractional Stefan problems have been introduced in the literature there have been no full numerical solutions offered. The object of this paper is to provide such solutions. In particular, numeral solutions for the fractional diffuse interface model in the sharp interface limit will be developed. Emphasis will be placed on comparing the numerical solutions with available closed form solutions. Obvious candidates for this task would be the solutions for the sharp interface Stefan problem derived by Liu and Xu [8] and Li et al. [9], and the solutions in the moisture transport limit ($c \to 0$) derived by Voller [5]. In light of the above discussion, however, care has to be taken because it is not to be expected that the fractional time sharp interface solutions will agree with the diffusive interface solution counterpart. To overcome this short fall we will introduce a new closed form solution for the fractional time derivative model of the diffuse moisture transport problem. A solution that explicitly exposes the disagreement between the fractional time sharp and diffuse interface models and is readily able to verify the fractional time form of the proposed numerical solution.

The paper is laid out as follows. In the next section sharp and diffusive interface (enthalpy) Stefan models are derived. Following, a brief primer on fractional calculus, the derivation of fractional Stefan problems is made, including discussion of appropriate closed from solutions. We next develop implicit time stepping numerical solutions of the diffusive interface form of the fractional Stefan model. It is shown that, in the sharp interface limit $\epsilon \to 0$, these solutions exhibit the expected time scale exponents and agree with a suite of available analytical solutions.

2. Stefan models

2.1. A problem statement

To streamline discussion and development we will focus on a one-phase, one-dimensional Stefan problem with constant thermal