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Mass/heat transfer through laminar boundary layer in axisymmetric microchannels with nonuniform cross section and fixed wall concentration/temperature

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ABSTRACT

We present a similarity solution for mass/heat transfer in laminar forced convection at high Peclet numbers. The classical boundary layer solution of the Graetz–Nusselt problem, valid for straight channels or pipes, is generalized to an axisymmetric microchannel with circular cross-section, whose radius $R(z)$ varies continuously along the axial coordinate z. The case of fixed wall concentration/temperature is analyzed.

The advection/diffusion transport problem is solved by taking into account both the tangential and normal velocity components (and their scaling behaviours as a function of the wall normal distance), in order to obtain an accurate description of the concentration/temperature profile in the boundary layer.

The analytical solution of the local Sherwood/Nusselt number is compared with finite elements numerical results for a truncated cone and a wavy sinusoidal channel.

1. Introduction

A correct estimation of heat and mass transfer coefficients is a powerful tool in the design of heat exchangers, mass transfer equipments and reactors, as well as micro-devices for chemical and biomedical applications [1-6]. Focusing on laminar forced convection of an incompressible fluid in a duct, the estimation of transport coefficients requires the solution of the classical Graetz–Nusselt problem [7,8]. Originally proposed for a sudden step change of the wall temperature at some positions along the duct and no axial diffusion [9,10], the Graetz–Nusselt problem is valid for both heat and mass transfer and it has been solved in transient and steady state [11], for Dirichlet and Neumann boundary conditions [12], non-Newtonian fluids [13], high viscous dissipation [14], boundary condition of continuity between two counterflow streams [15], axial diffusion [16,17], simultaneous heat and mass transfer [18–22]. The boundary layer problem, valid in the limit of $Pe \to \infty$, has been analytically solved in channels with constant cross-section for a variety of different cross-sections [23]. On the other hand, convection–diffusion transport in converging or diverging flows has been addressed for Taylor dispersion at low Reynolds number [24], but very few efforts have been devoted to the Graetz–Nusselt problem in converging–diverging channels. Non-parallel ducts have been indicated as a possible strategy to enhance heat transfer and numerical examples have been shown to corroborate the so-called “field synergy principle” [25]. Castelloes et al. [26] investigated heat transfer enhancement in converging–diverging channels in laminar flow conditions for $1 < Pe < 100$. The energy equation was solved using a hybrid numerical-analytical approach based on the Generalized Integral Transform Technique (GITT) in partial transformation mode for a transient formulation.

Recently, an analytical solution [27] has been proposed for the combined diffusive and convective mass transport from a surface film of arbitrary shape at a given uniform concentration to a pure solvent, flowing in the creeping regime through converging–diverging microchannels with slender rectangular cross-section. In [27] Adrover and Pedacchia clearly show that, close to the curved releasing boundary, both the tangential v_t and the normal velocity v_n components play a role in the mass transfer process, and their scaling behaviour as a function of wall the normal distance should be taken into account for an accurate description of the concentration profile in the boundary layer.

By following similar arguments, we present a similarity solution for mass/heat transfer in laminar forced convection at high Peclet numbers in axisymmetric microchannel with circular cross-section, whose radius $R(z)$ varies continuously along the axial coordinate z. Creeping flow conditions and fixed concentration/temperature at the channel wall are assumed. High values of Peclet number, together with low Reynolds numbers are often

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Nomenclature

List of symbols

c concentration
D diffusion coefficient
$g(z,Pe)$ rescaling function, Eq. (16)
h mass/heat transfer coefficient
$I(z)$ integral function, Eq. (21)
L_z channel length
$\mathbf{n}$ vector normal to the releasing wall at the point $(R_z(z),z)$
$Pe = v_R R_0/D$ cross-sectional Peclet number
$Pe_{\mathrm{eff}} = \alpha Pe$ effective Peclet number
Pe_l local Peclet number
$\tilde{r},\ \tilde{z}$ radial and axial dimensional coordinates
$r,\ z$ radial and axial dimensionless coordinates
$R_z(z)$ dimensionless cross-section radius depending on the axial position z
$R_z'(z) = dR_z/dz$ first order derivative of $R_z(z)$
$R_z''(z) = d^2R_z/dz^2$ second order derivative of $R_z(z)$
R_0 radius of the inlet section
$R = R(s) = R_z(z(s))$ dimensionless cross-section radius depending on the curvilinear abscissa s
$R'(s) = dR_z(z)/dz|_{z(s)}$ first order derivative of $R_z(z)$ evaluated at $z(s)$
$R''(s) = d^2R_z(z)/dz^2|_{z(s)}$ second order derivative of $R_z(z)$ evaluated at $z(s)$
s curvilinear abscissa
$Sh = h\ R_0/D$ Sherwood number
Sh_{app} approximate Sherwood number evaluated by neglecting the nornal convective term
$\mathbf{t}$ vector tangent to the releasing wall at the point $(R_z(z),z)$
T temperature
$v_n^0(s)$ prefactor of the quadratic term of the normal velocity component $v_n(\delta,s) \simeq v_n^0(s)\delta^2$
$v_t^0(s)$ prefactor of the linear term of the tangent velocity component $v_t(\delta,s) \simeq v_t^0(s)\delta$
$v_r,\ v_z$ dimensionless radial and axial velocity components
$v_n,\ v_t$ dimensionless normal and tangent velocity components
v_R average inlet axial velocity

Greek symbols

$\alpha = R_0/L_z$ channel aspect ratio
δ wall normal distance
$\eta = \delta g(s,Pe)$ similarity variable
ϕ dimensionless scalar field (concentration or temperature)

encountered in microfluidic applications, especially in connection with mass transport problems characterized by low diffusivity values, see e.g. micro-mixing devices [28–31], dispersion problems [32] and wide-bore chromatography [33,34].

2. Statement of the problem and numerical solutions

Let us consider an incompressible fluid moving in creeping flow conditions through an axisymmetric microchannel with circular cross-sections, whose radius varies along the axial coordinate.

Let $\tilde{r}$ and $\tilde{z}$ be the radial and axial coordinates, R_0 the radius of the inlet section, L_z the channel length and D the diffusion coefficient. Let ϕ be a scalar field representing a dimensionless concentration or temperature

$$\phi = \frac{c - c_{\mathrm{inlet}}}{c_{\mathrm{wall}} - c_{\mathrm{inlet}}} = \frac{T - T_{\mathrm{inlet}}}{T_{\mathrm{wall}} - T_{\mathrm{inlet}}}$$

In terms of dimensionless spatial coordinates $z = \tilde{z}/R_0,\ 0 \leqslant z \leqslant 1/\alpha = L_z/R_0$ and $r = \tilde{r}/R_0,\ 0 \leqslant r \leqslant R_z(z)$ (see Fig. 1) the steady-state convection–diffusion transport equation and boundary conditions (fixed wall concentration/temperature $c_{\mathrm{wall}}/T_{\mathrm{wall}}$ and Danckwerts inlet–outlet conditions) read as

$$-Pe\,v_r(r,z)\frac{\partial\phi}{\partial r} - Pe\,v_z(r,z)\frac{\partial\phi}{\partial z} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = 0 \qquad (1)$$

$$\left[Pe\,v_z\phi - \frac{\partial\phi}{\partial z}\right]_{z=0} = 0,\quad \phi(R_z(z),z) = 1,\quad \left.\frac{\partial\phi}{\partial r}\right|_{r=0} = 0,\quad \left.\frac{\partial\phi}{\partial z}\right|_{z=1} = 0 \qquad (2)$$

where $\alpha = R_0/L_z \ll 1$ is the aspect ratio, $Pe = v_R R_0/D$ is the cross-sectional Peclet number ($Pe = Re\ Sc$ or $Pe = Re\ Pr$) evaluated with respect to the average inlet axial velocity v_R. Let $v_r(r,z)$ and $v_z(r,z)$ be the dimensionless velocity components:

$$v_r(r,z) = \frac{2rR_z'}{\pi R_z^3}\left(1 - \left(\frac{r}{R_z}\right)^2\right) \qquad (3)$$

$$v_z(r,z) = \frac{2}{\pi R_z^2}\left(1 - \left(\frac{r}{R_z}\right)^2\right),\quad \int_0^{R_z} v_z(r,z)2\pi r dr = 1 \qquad (4)$$

where $R_z' = d\ R_z/d\ z$. The parabolic axial velocity profile $v_z(r,z)$ is evaluated from lubrication theory by enforcing unitary flow rate and no-slip boundary conditions. The radial velocity component $v_r(r,z)$ is obtained by enforcing the continuity equation in cylindrical coordinates [27]. Creeping flow conditions are assumed.

The local mass/heat transfer coefficient h can be expressed in terms of the Sherwood/Nusselt number $Sh = h\ R_0/D$, $Nu = h\ R_0/k$ evaluated from the gradient at the releasing wall as:

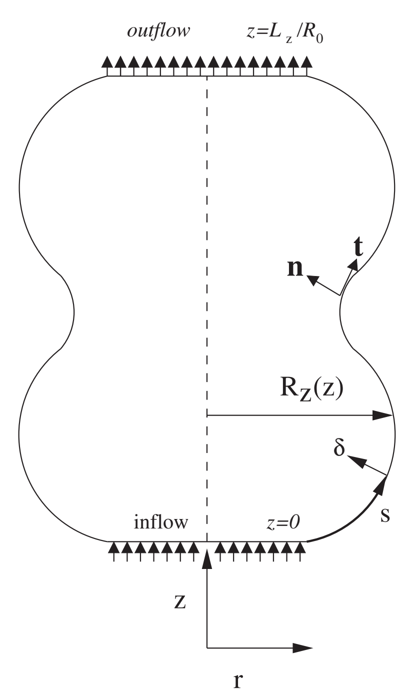

Fig. 1. Schematic representation of a channel longitudinal section at $\theta = 0$. r and z represent the dimensionless radial and axial coordinates. δ is the dimensionless wall normal distance. s is a curvilinear abscissa measured from the channel inlet section. $\mathbf{n}$ and $\mathbf{t}$ are the vectors normal and tangent to the releasing wall.