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Lie group analysis of flow and heat transfer over a stretching rotating disk

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ABSTRACT

The present paper discusses steady three dimensional flow and heat transfer of viscous fluid on a rotating disk stretching in radial direction. Using Lie group theory symmetries of the governing equations are calculated. Imposing restrictions from the boundary conditions it is shown that the similarity in the problem can be achieved for two types of radially stretching velocities namely; linear and power-law. Linear stretching has already been discussed in the literature; however power-law stretching is discussed here for the first time. Using new similarity transformations, the governing partial differential are transformed into a system of ordinary differential equations which are later treated both analytically and numerically. Exact analytical solutions are found for the case of pure stretching and the large stretching parameter case, for power-law stretching index $n = 3$. Numerical solutions are obtained for combined effects of stretching and rotation for all values of n using Keller box method. Comparison of numerical solution with the corresponding analytical solution (for $n = 3$) shows an excellent agreement. The quantities of physical interest, such as azimuthal and radial skin friction and also Nusselt number are presented and discussed physically.

1. Introduction

The study of flow field due to a rotating disk has found many applications in different fields of engineering and industry. A number of real processes can be undertaken using disk rotation such as: fans, turbines, centrifugal pumps, rotors, viscometers, spinning disk reactors and other rotating bodies. The history of rotating disk flows goes back to the celebrated paper by Von Karman [1] who initiated the study of incompressible viscous fluid over an infinite plane disk rotating with a uniform angular velocity. This model is further investigated by many researchers to provide analytical and numerical results for better understanding of the fluid behavior due to rotating disks.

The use of similarity transformations to convert governing Navier Stokes equations for axi-symmetric flow into a system of coupled nonlinear ordinary differential equations was originated by Von Karman [1] and the numerical results for these equations were presented by Cochran [2]. Millsaps and Pohlhausen [3] considered the effects of heat transfer over a rotating disk at a constant temperature. Awad [4] presented an asymptotic model to analyze the heat transfer phenomena over a rotating disk for large Prandtl numbers.

Finding exact solutions for the Navier–Stokes equations is of fundamental importance in understanding and development of fluid mechanics. Von Karman and Lin [5] gave the mathematical proof for the existence of exact solutions and Von Karman firstly presented the exact solutions for the flow over a rotating disk which is now a part of many classical textbooks [6–9]. The exact solutions for heat and mass transfer over a permeable rotating disk were presented by Turkyilmazoglu [10].

The flow due to stretching surfaces has important applications in manufacturing industries; especially in the extrusion of metals and plastics [11–13]. The exact analytical solution for steady linear stretching of a surface was given by Crane [14]. Wang [15] extended this problem to the three-dimensional case. Rashidi and Pour [16] found approximate analytical solutions for the flow and heat transfer over a stretching sheet using Homotopy Analysis Method. The steady flow over a rotating and stretching disk was initially proposed by Fang [17]. Recently, Fang and Zhang [18] studied the flow between two stretching disks. More recently, the combined effects of magnetohydrodynamic on radially stretching disk were analyzed by Turkyilmazoglu [19]. We observe that all of these studies were undertaken for linear radial stretching velocities. Gupta and Gupta [20] identified that stretching of the sheet

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may not necessarily be linear in real situations. The power-law stretching velocity was thus undertaken by Banks [21] and Ali [22].

Lie group analysis is a systematic way of finding the invariant or self-similar solutions of a system of partial differential equations. The method is capable of providing a deep insight into the underlying physical problems-described by partial differential equations. The applications of Lie group analysis are twofold: Producing a new solution from an existing solution or finding similarity solution of partial differential equation. The focus in the present paper is on the latter type of the application. Starting from the Sophus Lie (1842–1899), this technique is extensively used for finding solution of differential equations [23–25]. Jalil et al. [26] applied this method to find possible similarity transformation for the mixed convection flow over a stretching surface. They extended their work to flow of non-Newtonian fluids [27,28], by finding self similar solution of the governing equations, using Lie group analysis. Hamad et al. [29] investigated the combined effects of heat and mass transfer by Lie group analysis over a moving surface. Ferdows et al. [30] applied the method of one parameter continuous group theory to investigate mixed convection over horizontal moving porous flat plate. Recently, Ferdows et al. [31] used a special form of Lie group of transformation (scaling transformation) to study the convective effects of heat and mass transfer over a radiating stretching sheet.

The aim of this paper is to analyze the flow and heat transfer over a rotating disk that is stretching in the radial direction. This work is significant due to the following reasons (a) Finding of all possible similarity transformation for the problem using Lie group analysis. This leads us to discover the similarity transformations for linear stretching and power-law stretching. The linear stretching for rotating disk has already been available in the literature while the power-law stretching is worked out for the first time. (b) Using new similarity transformations the governing partial differential equations are transformed to self similar ordinary differential equations. Exact analytical solutions are found for pure stretching case and the case of large stretching parameter for power-law stretching index $n = 3$. (c) The numerical solutions for the combined effects of power-law stretching and rotation are obtained, for all n, employing the Keller box method [32]. To support the numerical results comparison between the exact analytical and numerical results, for pure stretching, is presented in the form of tables and figures. An excellent agreement is found between the two solutions. The effects of controlling parameters on the physical quantities are analyzed and discussed in detail.

2. Mathematical formulation

Let us consider a three dimensional laminar flow of a steady incompressible fluid over a rotating disk, which has a constant angular velocity Ω. The disk is stretching in radial direction with velocity $u_w(\tilde{r})$. The governing Navier–Stokes equations and energy equation with the corresponding boundary conditions for an axisymmetric flow and heat transfer in cylindrical coordinates are given by [33]:

$$\frac{1}{\tilde{r}}\frac{\partial(\tilde{r}\tilde{u})}{\partial\tilde{r}}+\frac{\partial\tilde{w}}{\partial\tilde{z}}=0, \tag{1}$$

$$\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{r}}+\tilde{w}\frac{\partial\tilde{u}}{\partial\tilde{z}}-\frac{\tilde{v}^2}{\tilde{r}}=-\frac{1}{\rho}\frac{\partial\tilde{p}}{\partial\tilde{r}}+\nu\left\{\frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}+\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\tilde{r}}+\frac{\partial^2\tilde{u}}{\partial\tilde{z}^2}-\frac{\tilde{u}}{\tilde{r}^2}\right\}, \tag{2}$$

$$\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{r}}+\tilde{w}\frac{\partial\tilde{v}}{\partial\tilde{z}}+\frac{\tilde{u}\tilde{v}}{\tilde{r}}=\nu\left\{\frac{\partial^2\tilde{v}}{\partial\tilde{r}^2}+\frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\tilde{r}}+\frac{\partial^2\tilde{v}}{\partial\tilde{z}^2}-\frac{\tilde{v}}{\tilde{r}^2}\right\}, \tag{3}$$

$$\tilde{u}\frac{\partial\tilde{w}}{\partial\tilde{r}}+\tilde{w}\frac{\partial\tilde{w}}{\partial\tilde{z}}=-\frac{1}{\rho}\frac{\partial\tilde{p}}{\partial\tilde{z}}+\nu\left\{\frac{\partial^2\tilde{w}}{\partial\tilde{r}^2}+\frac{1}{\tilde{r}}\frac{\partial\tilde{w}}{\partial\tilde{r}}+\frac{\partial^2\tilde{w}}{\partial\tilde{z}^2}\right\}, \tag{4}$$

$$\tilde{u}\frac{\partial\tilde{T}}{\partial\tilde{r}}+\tilde{w}\frac{\partial\tilde{T}}{\partial\tilde{z}}=\alpha_T\left\{\frac{\partial^2\tilde{T}}{\partial\tilde{r}^2}+\frac{1}{\tilde{r}}\frac{\partial\tilde{T}}{\partial\tilde{r}}+\frac{\partial^2\tilde{T}}{\partial\tilde{z}^2}\right\}, \tag{5}$$

$$\begin{aligned}&\tilde{z}=0; \quad \tilde{u}=\alpha\Omega\tilde{r}u_w(\tilde{r}/R),\ \tilde{v}=\Omega\tilde{r}v_w(\tilde{r}/R),\ \tilde{w}=0,\ \tilde{T}=\tilde{T}_w\\ &\tilde{z}\to\infty; \quad \tilde{u}=0,\ \tilde{v}=0,\ \tilde{T}=\tilde{T}_\infty.\end{aligned} \tag{6}$$

In the above equations $\tilde{u},\ \tilde{v}$ and $\tilde{w}$ are the components of velocity in $\tilde{r},\ \tilde{\theta}$ and $\tilde{z}$ directions, ρ is the fluid density, $\alpha_T(=k/\rho C_p)$ is the thermal diffusivity and $\tilde{p}$is the pressure. The parameter α is a constant known as disk stretching parameter.

3. Boundary layer equations

A pragmatic approach to find boundary layer equations is to introduce non-dimensional variables in the governing Eqs. (1)–(6). We consider the following non-dimensional variables for current problem

$$\begin{aligned}&r=\frac{\tilde{r}}{R},\quad z=\frac{\tilde{z}}{R}Re^{1/2},\quad u=\frac{\tilde{u}}{\Omega R},\quad v=\frac{\tilde{v}}{\Omega R},\quad w=\frac{\tilde{w}}{\Omega R}Re^{1/2},\\ &p=\frac{\tilde{p}}{\rho(\Omega R)^2},\quad T=\frac{\tilde{T}-\tilde{T}_\infty}{T_0},\end{aligned} \tag{7}$$

where $Re=\frac{\Omega R^2}{\nu}$ is Reynolds number, R is the reference length and T_0 is the reference temperature. It is noteworthy that the corresponding scales in the axial direction are smaller by a factor $Re^{-1/2}$, thus implicitly anticipating that $Re\gg 1$. Now the governing Eqs. (1)–(6) are converted to dimensionless form given as

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}=0, \tag{8}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{Re}\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right\}, \tag{9}$$

$$\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\frac{\partial^2 v}{\partial z^2}+\frac{1}{Re}\left\{\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right\}, \tag{10}$$

$$\frac{1}{Re}\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{1}{Re^2}\left\{\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+Re\frac{\partial^2 w}{\partial z^2}\right\}, \tag{11}$$

$$u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\frac{1}{Re\cdot \mathrm{Pr}}\left\{\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right\}+\frac{1}{\mathrm{Pr}}\frac{\partial^2 T}{\partial z^2}, \tag{12}$$

$$\begin{aligned}&z=0; \quad u=\alpha r u_w(r),\ v=r\,v_w(r),\ w=0,\ T=T_w\\ &z\to\infty; \quad u=0,\ v=0,\ T=0.\end{aligned} \tag{13}$$

where $\mathrm{Pr}=\nu/\alpha_T$ is the Prandtl number.

For high Reynolds number, i.e. $Re\to\infty$, the resulting boundary layer equations in dimensionless form are obtained as follows

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}=0, \tag{14}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\frac{\partial^2 u}{\partial z^2}, \tag{15}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\frac{\partial^2 v}{\partial z^2}, \tag{16}$$

$$0=-\frac{\partial p}{\partial z}, \tag{17}$$