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Technical Note

Exact solution for a Stefan problem with latent heat a power function of position

Yang Zhou [a,b,*], Yi-jiang Wang [a,b], Wan-kui Bu [a,c]

[a] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining & Technology, Xuzhou 221008, Jiangsu, PR China
[b] School of Mechanics & Civil Engineering, China University of Mining & Technology, Xuzhou 221116, Jiangsu, PR China
[c] Department of Mechanical and Electrical Engineering, Heze University, Heze 274015, Shandong, PR China

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ABSTRACT

A one-phase Stefan problem with latent heat a power function of position is investigated. The second kind of boundary condition is involved, and the surface heat flux is considered as a corresponding power function of time. The problem can be viewed as a special case of the shoreline movement problem under the conditions of nonlinear variation of ocean depth and a surface flux that varies as a power of time. An exact solution is constructed using the similarity transformation technique. Theoretical proof for the existence and the uniqueness of the exact solution is conducted. Solutions for some special cases presented in the literature are recovered. In the end, computational examples of the exact solution are presented, and the results can be used to verify the accuracy of general numerical phase change algorithms.

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1. Introduction

There are many problems that involve a moving boundary in industrial processes, and this type of problem is usually named as the Stefan problem (or the moving boundary problem). Many analytical and numerical solutions for these problems can be found in the monographs [1–5].

Recently, a special type of Stefan problem with space-dependent latent heat attracts much attention. This type of problem generally arises from the study of the shoreline movement. Numerical solutions can be found in [6–7]. Voller et al. [8] presented an exact solution for a one-phase Stefan problem with linearly distributed latent heat. Lorenzo-Trueba et al. [9] generalized Voller's problem by considering the nonlinearity of the diffusivity and the appearance of two moving boundaries; they presented analytical solutions for simple cases and numerical solutions for the general condition. Salva et al. [10] extended Voller's solution to a two-phase Stefan problem, and they also considered the linearly distributed latent heat.

Motivated by these works, we consider a one-phase Stefan problem with latent heat a power function of position. Mathematical equations for this problem are given by

$$\frac{\partial T}{\partial t} = v\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < s(t),\ t > 0, \tag{1}$$

$$T|_{x=s(t)} = 0, \quad t > 0 \tag{2}$$

$$k\frac{\partial T}{\partial x}\bigg|_{x=0} = -ct^{(n-1)/2}, \quad t > 0, \tag{3}$$

$$k\frac{\partial T}{\partial x}\bigg|_{x=s(t)} = -\gamma s(t)^n s'(t), \quad t > 0 \tag{4}$$

where T is the temperature, x is the position coordinate, t is the time coordinate, $s(t)$ is the moving interface, v is the thermal diffusion coefficient, k is the thermal conductivity, $ct^{(n-1)/2}$ is the time-varying surface heat flux ($c > 0$ for melting, $c < 0$ for freezing), γx^n is the variable latent heat per unit volume ($\gamma > 0$ for melting, $\gamma < 0$ for freezing), n is an arbitrary non-negative integer, and the phase-transition temperature is zero.

The Stefan problem described by Eqs. 1,2,3,4can be viewed as a special case of the shoreline movement problem under the conditions of nonlinear variation of ocean depth and a surface flux that varies as a power of time. In this problem, time-dependent surface flux of the form (3) is considered so that a similarity solution can be obtained; this is similar to Lombardi & Tarzia [11], in which the authors also uses the time-dependent surface flux in arriving a closed-form solution.

The main objective of this paper is to obtain an exact solution for the one-phase Stefan problem presented by Eqs. 1,2,3,4.

* Corresponding author at: State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining & Technology, Xuzhou 221008, Jiangsu, PR China. Tel.: +86 516 83995078.
E-mail address: tod2006@126.com (Y. Zhou).

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Section 2 constructs an exact solution using the similarity transformation technique, proves the existence and the uniqueness of the solution, and recovers solutions of some special cases. Section 3 presents computational examples of the exact solution, followed by Section 4, with some conclusions.

2. Exact solution

2.1. Solution procedure

Using the similarity transformation

$$f(\eta)=\frac{T(x,t)}{t^{n/2}}, \text{with } \eta=\frac{x}{2\sqrt{\nu t}} \tag{5}$$

The partial differential equation for $T(x, t)$ becomes the following ordinary differential equation for $f(\eta)$

$$f''+2\eta f'-2nf=0 \tag{6}$$

The solution for this ordinary differential equation can be written as

$$f(\eta)=Ai^n erfc(\eta)+Bi^n erfc(-\eta) \tag{7}$$

where A and B are arbitrary real constants, $i^n erfc(\cdot)$ are the repeated integrals of the complementary error function [12] defined by

$$erfc(\varepsilon)=\frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty \exp(-x^2)dx, \quad erf(\varepsilon) = 1-erfc(\varepsilon), \quad i^0 erfc(\varepsilon)=erfc(\varepsilon) \tag{8}$$

$$i^{-1}erfc(\varepsilon)=\frac{2}{\sqrt{\pi}}\exp(-\varepsilon^2), \quad i^n erfc(\varepsilon)=\int_\varepsilon^\infty i^{n-1}erfc(x)dx \tag{9}$$

Therefore the solution for Eq. (1) can be written as

$$T(x,t)=t^{n/2}\left[Ai^n erfc\left(\frac{x}{\sqrt{4\nu t}}\right)+Bi^n erfc\left(-\frac{x}{\sqrt{4\nu t}}\right)\right] \tag{10}$$

The above solution can also be verified from page 52 of the Carslaw and Jaeger text book [13] and Zhou [14].

In order to satisfy Eqs. (2,3), the moving interface must be given in the following form

$$s(t)=2\lambda\sqrt{\nu t} \tag{11}$$

where λ is a constant to be determined.

From Eq. (2) and (10), one obtains

$$Ai^n erfc(\lambda)+Bi^n erfc(-\lambda)=0 \tag{12}$$

From Eq. (3) and (10), one obtains

$$k[-Ai^{n-1}erfc(0)+Bi^{n-1}erfc(0)]=-2c\sqrt{\nu} \tag{13}$$

The coefficients A, B can be expressed by

$$A=\frac{c\sqrt{\nu}\cdot i^n erfc(-\lambda)}{ki_{0(n-1)}E_n(\lambda)} \tag{14}$$

$$B=-\frac{c\sqrt{\nu}\cdot i^n erfc(\lambda)}{ki_{0(n-1)}E_n(\lambda)} \tag{15}$$

where

$$i_{0(n-1)}=i^{n-1}erfc(0)=[2^{n-1}\Gamma[(n+1)/2]]^{-1}, \Gamma(\cdot) \tag{16}$$

is the gamma function

$$E_n(\lambda)=[i^n erfc(\lambda)+i^n erfc(-\lambda)]/2 \tag{17}$$

The equation for the coefficient λ can be constructed from Eq. (4)

$$\frac{\gamma 2^{n+1}(\sqrt{\nu})^{n+1}i_{0(n-1)}\lambda^{n+1}}{c}=\frac{i^n erfc(-\lambda)i^{n-1}erfc(\lambda)+i^n erfc(\lambda)i^{n-1}erfc(-\lambda)}{E_n(\lambda)} \tag{18}$$

Once λ is determined from Eq. (18), the exact solution for the one-phase Stefan problem presented by Eqs. 1,2,3,4 can be obtained from Eqs. (10), (11), (14), (15).

2.2. Existence and uniqueness

The exact solution for Eqs. 1,2,3,4 is constructed in Section 2.1. However, there remains the problem for the existence and the uniqueness of λ, and this problem can be solved by analyzing the monotonicity of the functions at the two sides of Eq. (18).

For the case n = 0, Eq. (18) becomes much simpler (Eq. (31) in Section 2.3); the existence and the uniqueness of λ can be proved easily. For the cases n $\geqslant$ 1, the proof is given below.

For convenience, we denote the left hand side of Eq. (18) as $L(\lambda)$, and the right hand side of Eq. (18) as $R(\lambda)$.For $L(\cdot)$, it is easy to prove that

$$L(0)=0, L(+\infty)=+\infty, \quad L'(\lambda)>0, \ \forall\lambda>0 \tag{19}$$

For $R(\lambda)$,

$$R'(\lambda)=[i^n erfc(\lambda)i^{n-2}erfc(-\lambda)-i^n erfc(-\lambda)i^{n-2}erfc(\lambda)]/E_n(\lambda) - R_1(\lambda)F_{n-1}(\lambda)/E_n^2(\lambda) \tag{20}$$

where

$$R_1(\lambda) \text{is the numerator of } R(\lambda), \tag{21}$$

$$F_{n-1}(\lambda)=dE_n(\lambda)/d\lambda=[i^{n-1}erfc(-\lambda)-i^{n-1}erfc(\lambda)]/2>0, \ \forall\lambda>0 \tag{22}$$

The recurrence formula for the repeated integrals of the complementary error function is

$$i^{n-2}erfc(\varepsilon)-2\varepsilon i^{n-1}erfc(\varepsilon)-2ni^n erfc(\varepsilon)=0 \tag{23}$$

Using the recurrence formula, Eq. (20) can be transformed to

$$R'(\lambda)=-\frac{\lambda}{n}[i^{n-2}erfc(-\lambda)i^{n-1}erfc(\lambda) + i^{n-1}erfc(-\lambda)i^{n-2}erfc(\lambda)]/E_n(\lambda)-R_1(\lambda)F_{n-1}(\lambda)/E_n^2(\lambda) < 0 \tag{24}$$

For λ = 0,

$$R(0)=2i_{0(n-1)}=2\cdot[2^{n-1}\Gamma[(n+1)/2])]^{-1}>0 \tag{25}$$

Since

$$i^n erfc(\infty)=0, \ \forall n\geq 0; \quad i^n erfc(-\infty)=+\infty, \ \forall n\geq 1 \tag{26}$$

$$\lim_{\lambda\to\infty}\frac{i^{n-1}erfc(-\lambda)}{i^n erfc(-\lambda)}=\lim_{\lambda\to\infty}\frac{erfc(-\lambda)}{ierfc(-\lambda)}=\frac{2}{\infty}=0, \ \forall n\geq 1 \tag{27}$$

We have

$$\lim_{\lambda\to\infty}R(\lambda)=\lim_{\lambda\to\infty}2\cdot i^n erfc(\lambda)\frac{i^{n-1}erfc(-\lambda)}{i^n erfc(-\lambda)}=0, \ \forall n\geq 1 \tag{28}$$

Eqs. (24), (25), (28) indicate that the right hand side of Eq. (18) is a strictly decreasing function, and it decreases from $2i_{0(n-1)}$ to 0 with λ increasing from 0 to ∞; Eq. (19) indicates that the left hand side of Eq. (18) is a strictly increasing function, and it increases from 0 to ∞ with λ increasing from 0 to ∞. Therefore there exists unique positive solution for Eq. (18).

2.3. Special cases

From references [13–14], we know that Eq. (10) is a solution of Eq. (1). By substituting Eqs. (10),(11) into Eqs. (2),(3),(4), the

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