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Analytical solution to heat conduction in finite hollow composite cylinders with a general boundary condition

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ABSTRACT

This paper presents a set of classical analytical solutions to heat conduction in a two-layer composite hollow cylindrical medium, which are derived by the method of Laplace transform. The subjected boundary conditions are general and included various combinations of constant temperature, constant flux, zero flux, or convection boundary condition at either surface. The new solution can reduce to Jaeger's solution to the problems subject to constant-temperature boundary conditions, verifying that our solution is an extension version of his solution. Moreover, the solutions subject to a constant flux and a constant temperature are used to evaluate short-time accuracy of a composite-medium line-source solution. Comparison of these two solutions indicates that the temperature response is always delayed as a result of the line-source assumption. An expression for estimating the minimum threshold, beyond which the line-source solution is acceptable, is suggested for engineering applications.

1. Introduction

Transient heat conduction in a composite medium is a classic problem of heat conduction. Composite materials are media composed of layers of media having different thicknesses and thermal properties. Since these multilayer materials have added advantages of combined properties, such as higher ratio of strength to weight, plasticity, low cost, etc., nowadays they are extensively used in numerous science and engineering applications, for instance, brake and friction systems, heat exchangers, electrical applications, and biomaterials [1–4]. They do, however, cause additional complications in the thermal analysis [5–7].

Various mathematical methods are available for solving heat conduction in a composite medium, including Laplace transform method [5–7], orthogonal expansion technique [8–11], Green's function approach [12], line heat-source method [13–15], and integral transform technique [16–18], etc. The Laplace transform method may be the best for solving one-dimensional transient problems. In their classical book [5], Carslaw and Jaeger summarized many such solutions for composite media. For multi-dimensional problems, the combined methods of Laplace transform and other techniques such as integral transform and separation of variables may be useful [5,19].

Recently, rapid advances of computers and software have generated growing interest in solving more general problems in multilayered materials, together with various complexities [12,19–24], for example, involving the one-dimensional [20] or two-dimensional n-layer materials [21], time-dependent boundary conditions [23], and n-layer orthotropic laminates [24]. These more general problems usually involve computation of higher-order matrix determinants [19], recurring relations for determining coefficients occurred in the eigenfunctions [20–23], and even numerical schemes for the Laplace inversion theorem [24]. In fact, these "analytical" solutions are only formal solutions, and they are not fully exact or explicit solutions in the classical sense. Not surprisingly, to the degree that these complications are involved, these solutions are purely formal and their implementation relies heavily on advanced commercial computational software.

In this paper, a set of classical explicit (not formal) analytical solutions for a two-layer composite hollow cylindrical medium with general inhomogeneous boundary conditions are derived. The boundary conditions are very general, including various combinations of the first-, the second-and the third-kind boundary conditions; it is shown that the analytical solution can reduce to some reported specific solutions if proper values of the coefficients in the general boundary conditions are specified. From a practical perspective, the fully-explicit solution is a good choice for validating the formal or semi-analytical solutions for more complicated problems. To the best of the authors' knowledge, the derived solutions here are new and contribute to the field of heat conduction. Another objective of this paper is to verify short-time performance

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of a new developed line-source solution for composite cylindrical media [14,15]. A simplified solution, subjected to a constant heat flux and a constant temperature, can serve as this verification.

2. Problem statement and solution

The problem studied here is the heat conduction in a finite hollow composite cylinder (Fig. 1). The region $r_1 \leqslant r < r_2$ is of one substance and region $r_2 \leqslant r < r_3$ of another. Initial temperatures are assumed to be zero for both regions; the governing equations are

$$\frac{1}{a_i}\frac{\partial\theta_i}{\partial t} = \frac{\partial^2\theta_i}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_i}{\partial r} \quad (r_i \leqslant r < r_{i+1}), \quad i = 1,2 \tag{1}$$

where a_1, a_2 denote thermal diffusivities of the media. The initial conditions are

$$t = 0, \qquad \theta_i = 0, \quad i = 1,2 \tag{2a}$$

The boundary conditions are

$$r = r_1, \quad h_1\frac{\partial\theta_1}{\partial r} - h_2\theta_1 = h_3 \tag{2b}$$

$$r = r_2, \quad k_1\frac{\partial\theta_1}{\partial r} = k_2\frac{\partial\theta_2}{\partial r} \tag{2c}$$

$$r = r_2, \quad \theta_1 = \theta_2 \tag{2d}$$

$$r = r_3, \quad h_1'\frac{\partial\theta_2}{\partial r} + h_2'\theta_2 = h_3' \tag{2e}$$

Here, h_1, h_2, h_1', h_2' are constants which may be positive or zero provided both of h_1 and h_2, or h_1' and h_2' do not vanish; h_3 and h_3' are time-independent constants. These boundary conditions include all combinations of constant temperature, constant flux, zero flux, or convection boundary condition at either surface.

For convenience, the following derivation uses functions suggested by Jaeger [25]:

$$D(x,y) = I_0(x)K_0(y) - K_0(x)I_0(y) \tag{3}$$

$$D_{r,s}(x,y) = \frac{\partial^{r+s}D}{\partial x^r\partial y^s} \tag{4}$$

$$C(x,y) = J_0(x)Y_0(y) - Y_0(x)J_0(y) \tag{5}$$

$$C_{r,s}(x,y) = \frac{\partial^{r+s}C}{\partial x^r\partial y^s} \tag{6}$$

Here, I_n and K_n denote the modified Bessel functions of order n, J_n and Y_n denote the Bessel functions of the first and the second kind of order n, respectively. The functions defined by Eqs. (3)–(6) are connected by the relations

$$D(ix, iy) = -\frac{\pi}{2}C(x,y) \tag{7}$$

$$i^{r+s}D_{r,s}(ix, iy) = -\frac{\pi}{2}C_{r,s}(x,y) \tag{8}$$

In this paper i is used for $i^2 = -1$.

The heat conduction problem defined by Eq. (1) and (2) can be solved by the Laplace transform method, and the key steps of this solution procedure are listed in the Appendix A; the final result is

$$\theta_1 = \frac{k_2r_3h_3'(h_1 + r_1h_2\ln r/r_1) + r_1h_3\left[r_3h_2'(k_2\ln r/r_2 - k_1\ln r_3/r_2) - k_1h_1'\right]}{k_2h_1h_2'r_3 + k_1h_1'h_2r_1 - h_2h_2'r_1r_3(k_2\ln r_1/r_2 + k_1\ln r_2/r_3)} - \pi k_2\sum_{n=1}^{\infty}\frac{\delta_1\delta_2\exp(-a_1\alpha_n^2t)}{C(r_2\alpha_n, r_1\alpha_n)C(r_2\kappa\alpha_n, r_3\kappa\alpha_n)F(\alpha_n)}$$
$$\times\left[(\delta_2k_1h_3 + k_2h_3'\delta_1)C(r_2\kappa\alpha_n, r_3\kappa\alpha_n)C(r\alpha_n, r_1\alpha_n) + \frac{\pi}{2}h_3k_2r_2\varepsilon_1C(r\alpha_n, r_2\alpha_n)\right] \tag{9}$$

$$\theta_2 = \frac{r_1r_3\left[k_1h_3h_2'\ln r/r_3 + h_2h_3'(k_1\ln r/r_2 - k_2\ln r_1/r_2)\right] + k_2h_1h_3'r_3 - k_1h_3h_1'r_1}{k_2h_1h_2'r_3 - h_2h_2'r_1r_3(k_2\ln r_1/r_2 + k_1\ln r_2/r_3) + k_1h_1'h_2r_1} - \pi k_2\sum_{n=1}^{\infty}\frac{\delta_1\delta_2\exp\left(-a_1\alpha_n^2t\right)}{C(r_2\alpha_n, r_1\alpha_n)C(r_2\kappa\alpha_n, r_3\kappa\alpha_n)F(\alpha_n)}$$
$$\times\left[\frac{\pi}{2}k_1h_3'r_2\varepsilon_2C(r\kappa\alpha_n, r_2\kappa\alpha_n) + (k_1h_3\delta_2 + k_2h_3'\delta_1)C(r_2\alpha_n, r_1\alpha_n)C(r\kappa\alpha_n, r_3\kappa\alpha_n)\right] \tag{10}$$

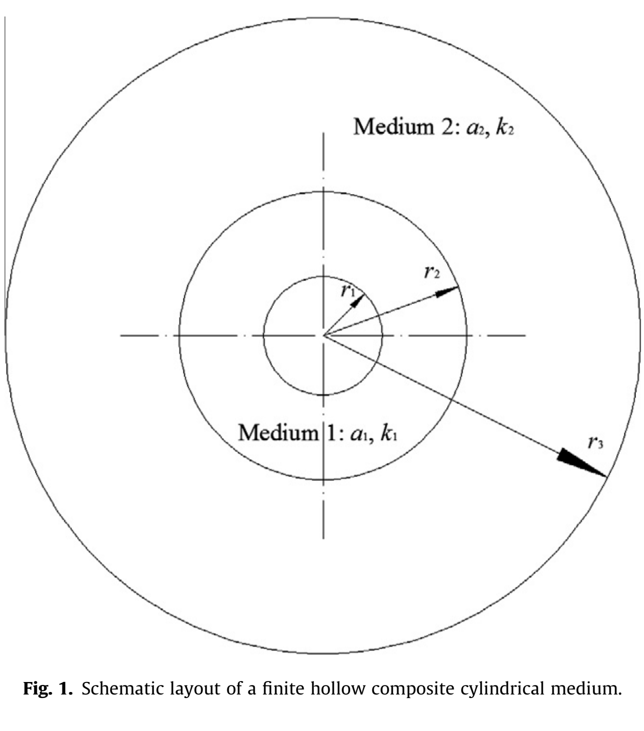

Fig. 1. Schematic layout of a finite hollow composite cylindrical medium.

where $\kappa = \sqrt{a_1/a_2}$ is a dimensionless ratio; $F(\alpha_n)$ is defined in the Appendix A by Eq. (A.26); and α_n are the roots of Eq. (11):

$$k_2\kappa\delta_1\delta_{21} + k_1\delta_2\delta_{11} = 0 \tag{11}$$

In Eqs. (9)–(11), the following definitions of variables are used:

$$\delta_1 = h_2C(r_1\alpha_n, r_2\alpha_n) - h_1\alpha_nC_{1,0}(r_1\alpha_n, r_2\alpha_n) \tag{12}$$

$$\delta_{11} = h_1\alpha_nC_{1,1}(r_2\alpha_n, r_1\alpha_n) - h_2C_{1,0}(r_2\alpha_n, r_1\alpha_n) \tag{13}$$

$$\delta_2 = h_1'\kappa\alpha_nC_{1,0}(r_3\kappa\alpha_n, r_2\kappa\alpha_n) + h_2'C(r_3\kappa\alpha_n, r_2\kappa\alpha_n) \tag{14}$$

$$\delta_{21} = h_1'\kappa\alpha_nC_{1,1}(r_2\kappa\alpha_n, r_3\kappa\alpha_n) + h_2'C_{1,0}(r_2\kappa\alpha_n, r_3\kappa\alpha_n) \tag{15}$$

$$\varepsilon_1 = C(\kappa r_2\alpha_n, \kappa r_3\alpha_n)\left[\frac{k_1}{k_2}\alpha_n\delta_2C_{1,0}(r_2\alpha_n, r_1\alpha_n) - \frac{4h_3'h_1}{\pi^2h_3r_1r_2}\right] - C(r_2\alpha_n, r_1\alpha_n)\left[\kappa\alpha_n\delta_2C_{1,0}(\kappa r_2\alpha_n, \kappa r_3\alpha_n) - \frac{4h_1'}{\pi^2r_2r_3}\right] \tag{16}$$

$$\varepsilon_2 = C(r_2\alpha_n, r_1\alpha_n)\left[\frac{k_2}{k_1}\kappa\alpha_n\delta_1C_{1,0}(r_2\kappa\alpha_n, r_3\kappa\alpha_n) + \frac{4h_3h_1'}{\pi^2h_3'r_2r_3}\right] - C(\kappa r_2\alpha_n, \kappa r_3\alpha_n)\left[\alpha_n\delta_1C_{1,0}(r_2\alpha_n, r_1\alpha_n) + \frac{4h_1}{\pi^2r_1r_2}\right] \tag{17}$$