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Heat conduction in a semi-infinite medium with a spherical inhomogeneity and time-periodic boundary temperature

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ABSTRACT

We solve the problem of heat conduction in a homogeneous media below a planar boundary subjected to time-periodic temperature (of frequency ω), in the presence of a spherical inhomogeneity (of radius R), whose center is at distance $d > R$ from the boundary. In the absence of the sphere, the well known one dimensional solution can be regarded as an oscillating thermal boundary layer of displacement thickness $\delta = \sqrt{2\alpha/\omega}$, where α is the heat diffusivity. The general solution depends on four dimensionless parameters: d/R, δ/R, the heat conductivity ratio κ and the heat capacity ratio C. An analytical solution is derived as an infinite series of Bessel functions, which converges quickly. The results are illustrated and analyzed for a given accuracy and for a few values of the governing parameters. The general solution can be simplified considerably for asymptotic values of the parameters. A first approximation, obtained for $R/d \ll 1$, pertains to an unbounded domain. A further approximate solution, for $R/\delta \ll 1$, while κ and C are fixed, can be regarded as pertaining to a quasi-steady regime, and is similar in structure to Maxwell's solution for steady state. However, its accuracy deteriorates for $\kappa \ll 1$, and a solution, coined as the insulated sphere approximation, is derived for this case. Comparison with the exact solution shows that these approximations are accurate for a wide range of parameter values. Besides providing insight, they can be employed for solving in a simple manner more complex problems, e.g. effective properties of a heterogeneous medium made of an ensemble of spherical inclusions.

1. Introduction

We consider here the problem of heat conduction into the half space, below a planar boundary subjected to time-periodic temperature. The medium of constant properties (conductivity, diffusivity) contains a spherical inclusion of different constant properties. While the solutions for a homogeneous medium or an isolated sphere subjected to given time dependent temperature on its boundary are well documented ([1] – Chapters 7, 8 and 11), much less is known for configurations similar to the present one. There is vast literature on the steady state problem (applicable to similar processes like electrical conduction, diffusion and flow through porous media) subjected to a uniform field at infinity, starting with the classical Maxwell solution [2]. Among the few previous works on the unsteady problem, the recent studies [3,4] are most relevant to the one pursued here. Though we follow their general approach of expanding the solution of the governing Helmholtz equation in a series of eigenfunctions, we explicitly derive the solution for the half space with periodic temperature variations, whereas they provide a general approach for an infinite domain solely. Furthermore, we present here for the first time a full analytical solution to the above problem together with simple approximate solutions which lend themselves to physical interpretation. Such solutions may be extremely useful when solving more complex problems.

The recent article of [5] also addresses a similar time dependent problem for a spherical inclusion, yet it is limited to a highly conducting sphere with only radial dependence of temperature and it assumes heat generation. Another article which solves a mathematical problem similar to that of this work and presents some approximations as well is [6]. Nevertheless, the problem considered there is of diffusive interaction between two ideal sinks, which is not time periodic and has simpler boundary conditions.

The applications of primary interest to us are of a geophysical nature, e.g. the impact of heterogeneity upon heat flow through the earth crust under periodic diurnal or annual temperature variations or the related problem of fluid flow through elastic porous formations in the soil. However, other potential applications like heat exchange between blood tissue and embedded blood vessels [7], the hydrocooling of fruits or vegetables of spherical shape [8] and experimental methods for specifying the thermal diffusivity of materials [9], can also be envisaged. Similarly, the present solution may constitute the first step toward determining equivalent properties of heterogeneous media under unsteady conditions.

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Thus, we consider the problem addressed here as of fundamental nature in view of the possible ramifications of the solution.

The plan of the paper is as follows: the problem is stated mathematically in Section 2 and its exact solution is derived subsequently in Section 3, expressed in terms of the various dimensionless parameters of the problem. Two simple solutions coined here as complex Maxwell (CM) and the insulated sphere approximations (ISA) are derived in Section 4 and their accuracy is determined by comparison against the exact one. The impact of the spherical inhomogeneity upon the surface heat flux, which is perpendicular to the planar boundary, is examined in Section 5. The paper is concluded with a summary and discussion in Section 6.

2. Mathematical statement

We consider a semi-infinite domain subjected to periodic temperature variation and enclosing a spherical body of radius R (Fig. 1). The boundary temperature is assumed to be constant in space and to vary sinusoidally in time. We further assume a medium of constant ambient thermal diffusivity, α_{ex}, with a spherical inhomogeneity of radius R and diffusivity α_{in} located at a depth $d \geqslant R$ from the surface ($\alpha_{ex/in} = K_{ex/in}/\left(\rho_{ex/in} \cdot C_p^{ex/in}\right)$, where K is the heat conductivity, ρ the density and C_p the specific heat of the two media). For convenience we use both cartesian and spherical coordinates defined by $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$ and $z = r\cos\theta$, taking the origin of the axis to be at the sphere center (see Fig. 1). The governing equations for the temperature of the sphere interior (denoted "in") and the exterior temperature (denoted "ex") are

$$\frac{\partial T_{ex/in}}{\partial t} = \alpha_{ex/in} \cdot \nabla^2 T_{ex/in} \quad z \leqslant d \tag{1}$$

with the boundary conditions

$$T_{ex} = T_0 \cos(\omega t), \quad z = d \tag{2a}$$

$$T_{ex} = T_{hom}, \quad r \to \infty \tag{2b}$$

where ω is the prescribed frequency of the surface temperature and T_{hom} is the solution for the temperature field in the homogeneous medium of diffusivity α_{ex}. The usual conditions for continuity of temperature and heat flux across the spherical boundary are as follows:

$$T_{ex} = T_{in}, \quad r = R \tag{3a}$$

$$\frac{\partial T_{ex}}{\partial r} = \kappa \frac{\partial T_{in}}{\partial r}, \quad r = R \tag{3b}$$

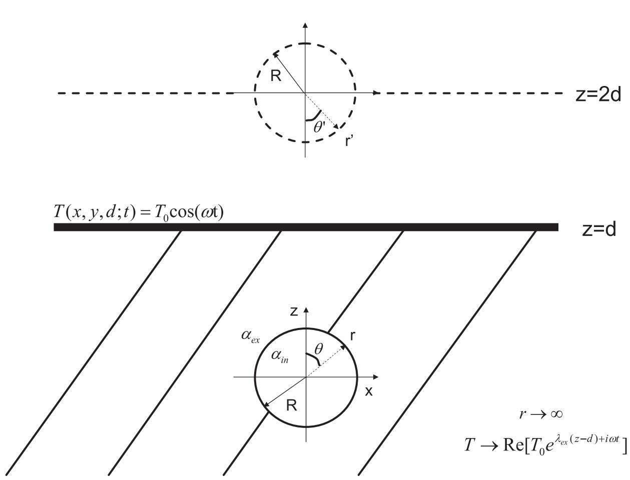

Fig. 1. The setup of the problem as formulated by (1) and (2a,b). The sphere and its image are denoted by a solid and dashed circle respectively and a planar boundary is located at a distance d above the sphere center.

where $\kappa = K_{in}/K_{ex}$.

We reformulate the problem with the aid of dimensionless variables, to simplify its parameter dependence. Thus, variables are normalized by using R as length, ω^{-1} as time and T_0 as temperature scales. Hence, the dimensionless variables are defined by x/R, y/R, z/R, r/R, d/R, T/T_0, $\alpha_{ex/in}/\omega R^2$, ωt and for convenience we maintain the original notation for these variables in the following. The boundary condition (2a) implies the solution is of the form $T = Re[\widehat{T}e^{it}]$ allowing us to solve for the time independent variable, $\widehat{T}$. The following modified Helmholtz equations are then derived from (1) for the complex temperatures $\widehat{T}_{ex/in}$

$$\nabla^2 \widehat{T}_{ex/in} = \lambda^2_{ex/in} \cdot \widehat{T}_{ex/in} \tag{4}$$

where $\lambda_{ex/in} = (i+1)/\sqrt{2\alpha_{ex/in}}$. The boundary conditions (2a,b) satisfied by $\widehat{T}_{ex}$ are consequently

$$\widehat{T}_{ex}(x, y, d) = 1, \quad \widehat{T}_{ex} \to \widehat{T}_{hom} \quad (\text{for } r \to \infty) \tag{5}$$

The solution of the heat flow problem in a homogeneous media was presented and analyzed in [1] (Section 2.6) and is given by

$$\begin{aligned} &\widehat{T}_{hom} = e^{\lambda_{ex}(z-d)}, \quad \text{i.e. } |\widehat{T}_{hom}| = e^{(z-d)/\sqrt{2\alpha_{ex}}} = e^{(z-d)/\delta} \\ &\arg(\widehat{T}_{hom}) = (z-d)/\sqrt{2\alpha_{ex}} = (z-d)/\delta \end{aligned} \tag{6}$$

where $\delta = \int_{-\infty}^{d} |\widehat{T}_{hom}| dz = \sqrt{2\alpha_{ext}}$ is the displacement thickness of the thermal boundary layer defined by $|\widehat{T}_{hom}|$, a convenient length scale characterizing the depth of penetration into the medium.

It is convenient to recast the problem for T'_{ex}, the normalized perturbation temperature associated with the spherical inclusion, defined by

$$\widehat{T}_{ex} = e^{-\lambda_{ex}d} T'_{ex} + \widehat{T}_{hom} = e^{-\lambda_{ex}d}\left(T'_{ex} + e^{\lambda_{ex}z}\right), \quad r \geqslant 1 \tag{7}$$

while T'_{in} is defined by

$$\widehat{T}_{in} = e^{-\lambda_{ex}d} T'_{in}, \quad r \leqslant 1 \tag{8}$$

Thus, it is found from (4), (5) and (3a,b) that the complete set of equations and boundary conditions satisfied by $T'_{ex/in}$ are as follows:

$$\nabla^2 T'_{ex} - \lambda^2_{ex} \cdot T'_{ex} = 0, \quad z \leqslant d, \ r \geqslant 1 \tag{9a}$$

$$\nabla^2 T'_{in} - \lambda^2_{in} \cdot T'_{in} = 0, \quad z \leqslant d, \ r \leqslant 1 \tag{9b}$$

$$T'_{ex} = 0, \quad z = d \tag{10a}$$

$$T'_{ex} = 0, \quad r \to \infty \tag{10b}$$

$$T'_{in} - T'_{ex} = e^{\lambda_{ex}z}, \quad r = 1 \tag{11a}$$

$$\kappa \frac{\partial T'_{in}}{\partial r} - \frac{\partial T'_{ex}}{\partial r} = \frac{\partial}{\partial r} e^{\lambda_{ex}z}, \quad r = 1 \tag{11b}$$

It is seen that for $\kappa = 1$ and $\alpha_{ex} = \alpha_{in}$ the solution of the system (9)–(11) is given by $T'_{ex} \equiv 0$, $T'_{in} = e^{\lambda_{ex}z}$, i.e. $\widehat{T} \equiv \widehat{T}_{\text{hom}}$.

Defining the ratio $C = \left(\rho_{ex}C_p^{ex}\right)/\left(\rho_{in}C_p^{in}\right)$ along with the relationships $\alpha_{ex} = \delta^2/2$ and $\alpha_{in} = \kappa C\delta^2/2$, renders the temperature fields as functions of four independent dimensionless parameters: κ, δ, C and d.

3. Exact solution

The general solution of (9a,b) for T' is given for the sphere exterior and interior, respectively, by [10]:

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} e^{im\varphi} P_n^m(\mu) \frac{k_n(\lambda_{ex} \cdot r)}{k_n(\lambda_{ex})}, \quad r \geqslant 1 \tag{12a}$$

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} e^{im\varphi} P_n^m(\mu) \frac{g_n(\lambda_{in} \cdot r)}{g_n(\lambda_{in})}, \quad r \leqslant 1 \tag{12b}$$