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A finite volume method for cylindrical heat conduction problems based on local analytical solution

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ABSTRACT

A new finite volume method for cylindrical heat conduction problems based on local analytical solution is proposed in this paper with detailed derivation. The calculation results of this new method are compared with the traditional second-order finite volume method. The newly proposed method is more accurate than conventional ones, even though the discretized expression of this proposed method is slightly more complex than the second-order central finite volume method, making it cost more calculation time on the same grids. Numerical result shows that the total CPU time of the new method is significantly less than conventional methods for achieving the same level of accuracy.

1. Introduction

Numerical methods, such as finite difference method, finite volume method, finite analytic method and finite element method, are used to solve heat conduction problems, among which the second-order central finite volume method is a most widely used one [1].

In order to obtain more accurate results in solving heat conduction problems, a lot of studies have been carried out to accurately determine the diffusion coefficient at the interface of the control volume, such as arithmetic mean, harmonic mean and integral mean interpolation method. Arithmetic mean [2,3], which corresponds to linear interpolation between two control nodes, is easy to handle as well as to program, making it widely used in the early days [1]. Patankar [4] proposed harmonic mean based on thermal resistance in series principle in 1978. Date [5] compared the performance of arithmetic mean with that of harmonic mean, and he highly recommended employing the latter. Harmonic mean has become a mainstream interpolation scheme because of its clear physical interpretation, especially suitable for composite medium. Voller and Swaminathan [6] presented an integral mean interpolation scheme based on Kirchhoff transformation. The scheme has higher accuracy than arithmetic mean and harmonic mean, but it needs numerical integration during the calculation, leading to a larger workload, especially when the integral of diffusion coefficient cannot be expressed analytically. Harmonic mean and integral mean are proposed to improve the precision and convergence rate.

During the Numerical Heat Transfer course given in China University of Petroleum-Beijing, the students were asked to compare the calculations of two-dimensional heat conductions on a Cartesian coordinate and a cylindrical coordinate. To our surprise, it is found that more grids are needed to obtain a grid-independent solution for a cylindrical case under the same boundary conditions and other conditions, especially when the ratio of the inner radius to the outer radius is small. Fig. 1 shows the computational domain. Fig. 2 and Table 1 show an example of the comparison of the relative error E, defined by $E = \sum_{N=1}^{N_{Grid}} \frac{(T_c - T_b)}{T_b} / N_{Grid} \times 100\%$ where T_c and T_b are respectively the computed temperatures and grid-independent solutions. It can be easily drawn that the slower convergence rate of cylindrical heat conduction is due to the conduction area decreases with the decrease of radius, making the heat flux related to radius. Without considering the influence of the radius when we used harmonic mean, the calculation precision becomes lower. In order to improve the convergence rate of cylindrical heat conduction, this paper presents a new finite volume method based on local analytical solution, acquiring higher precision while employing fewer nodes at the same time.

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Nomenclature

a_P, a_E, a_W, a_N, a_S coefficients in the discretized equation
b source term in the discretized equation
c_p heat capacity, J/(kg °C)
E relative error comparison, $E = \sum_{N=1}^{N_{Grid}} \frac{(T_c - T_b)}{T_b} / N_{Grid} \times 100\%$
Gr_f Grashof number, $Gr_f = g\beta(T_h - T_l)l^3/\nu^2$
g acceleration coefficient due to gravity, m/s^2
h_f convection heat transfer coefficient, W/(m^2 °C)
h height of domain, m
l length of domain, m
N_{Grid} total grid number
Pr_f Prandtl number, $Pr_f = \nu/\alpha$
q heat flux density, W/m^2
q_B heat flux density of boundary, W/m^2
r spatial coordinate
r_1 internal radius, m
r_2 external radius, m
r_3 external radius in Fig. 15, m
R non-dimensional coordinate in r direction, $R = (r - r_1)/(r_2 - r_1)$
R_T CPU time ratio of new scheme to central difference scheme
S heat source, W/m^3
S^* source term, $S^* = S + \frac{\partial}{\partial x}(\lambda \frac{\partial T}{\partial x})$, W/m^3
T temperature, °C
T_B temperature of the bottom wall, °C
T_f ambient temperature, °C
T_h high temperature, °C
T_l low temperature, °C
T_L temperature of the left wall, °C
T_R temperature of the right wall, °C
T_T temperature of the top wall, °C
x spatial coordinate
X non-dimensional coordinate in x direction, $X = x/l$

Greek symbols
α thermal diffusivity, m^2/s
β thermal expansion coefficient, °C^{-1}
λ thermal conductivity, W/(m °C)
λ_s thermal conductivity of solid in Fig. 15, W/(m °C)
λ_f thermal conductivity of fluid in Fig. 15, W/(m °C)
ν kinematic viscosity, m^2/s
τ time, s
ρ density, kg/m^3
Δx, Δr width of control volume in the x and r direction
δx, δr distance between adjacent nodes

Subscripts
car Cartesian coordinate
cyl cylindrical coordinate
e, w, n, s interfaces of the control volume P as shown in Fig. 3
$P, E, W, N, S, NE, SE, NW, SW$ grid points as shown in Fig. 3

2. Discretization of heat conduction equation in a cylindrical coordinate

Firstly, let us review discretization of a two-dimensional cylindrical heat conduction using the finite volume method. The steady-state heat conduction equation in a cylindrical coordinate can be written as follows:

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda \frac{\partial T}{\partial r}\right) + S = 0 \tag{1}$$

Integrating over the control volume P, as shown in Fig. 3, we can obtain:

$$\left(\lambda \frac{\partial T}{\partial x}\bigg|_e - \lambda \frac{\partial T}{\partial x}\bigg|_w\right)\Delta r_P + \frac{1}{r_P}\left(r\lambda \frac{\partial T}{\partial r}\bigg|_n - r\lambda \frac{\partial T}{\partial r}\bigg|_s\right)\Delta x_P + \Delta x_P \Delta r_P S_P = 0 \tag{2}$$

Discretized the first-order derivative by a second-order central difference scheme, Eq. (2) can be transformed to the expression below:

$$r_P\left(\lambda_e \frac{T_E - T_P}{(\delta x)_e} - \lambda_w \frac{T_P - T_W}{(\delta x)_w}\right)\Delta r_P + \left(r_n\lambda_n \frac{T_N - T_P}{(\delta r)_n} - r_s\lambda_s \frac{T_P - T_S}{(\delta r)_s}\right)\Delta x_P + r_P \Delta x_P \Delta r_P S_P = 0 \tag{3}$$

Rearranging Eq. (3), we can obtain the following equation:

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + b \tag{4}$$

where

$$a_P = a_W + a_E + a_S + a_N$$
$$a_W = \frac{r_P \lambda_w \Delta r_P}{(\delta x)_w}, \quad a_E = \frac{r_P \lambda_e \Delta r_P}{(\delta x)_e}, \quad a_S = \frac{r_s \lambda_s \Delta x_P}{(\delta r)_s}, \quad a_N = \frac{r_n \lambda_n \Delta x_P}{(\delta r)_n}$$
$$b = r_P \Delta x_P \Delta r_P S_P$$

Thermal conductivity on the interface λ_w, λ_e, λ_n, λ_s can be calculated by arithmetic mean or harmonic mean. Take the north interface as an example, when arithmetic mean is adopted, then:

$$\lambda_n = \frac{(r_N - r_n)\lambda_P + (r_n - r_P)\lambda_N}{r_N - r_P} \tag{5}$$

When harmonic mean is employed:

$$\lambda_n = \frac{(r_N - r_P)\lambda_P \lambda_N}{(r_n - r_P)\lambda_N + (r_N - r_n)\lambda_P} \tag{6}$$

This paper aims at how to reduce discrete error as well as to improve convergence rate to get grid-independent solution. In this regard, the method employed in this paper, which based on local analytical solution, intends to improve the accuracy of radial heat flux q_n, q_s (first-order derivative $\lambda \frac{\partial T}{\partial r}|_n, \lambda \frac{\partial T}{\partial r}|_s$) in Eq. (2). To obtain the analytical expression of q_n, q_s, we can rewrite Eq. (1) as follows:

$$\frac{\partial}{\partial r}\left(r\lambda \frac{\partial T}{\partial r}\right) + rS^* = 0 \tag{7}$$

where

$$S^* = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + S$$

and integrate Eq. (7) over interval $P - n$ and $n - N$ shown in Fig. 3.

In the control volume P and N, we assume thermal conductivities are constant λ_P and λ_N while source terms are S_P^* and S_N^*, respectively. Based on the above assumptions, the temperature expression over interval $P - n$ and $n - N$ can be obtained below,Interval $P - n$:

$$T = -\frac{1}{4}\frac{S_P^*}{\lambda_P} r^2 + C_1 \ln r + C_2 \tag{8}$$

Interval $n - N$:

$$T = -\frac{1}{4}\frac{S_N^*}{\lambda_N} r^2 + C_3 \ln r + C_4 \tag{9}$$

Thus the temperatures at r_P, r_N can be written as follows: