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Technical Note

Convection in a porous layer with a surface reaction ☆

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ABSTRACT

We develop a problem recently studied by Postelnicu in which a horizontal layer of porous material is subject to a chemical reaction at the base of the layer. The porous medium is of Darcy type and the layer is saturated with a non-isothermal liquid containing a concentration of the reacting chemical. The equations are developed and then it is shown that if the reaction parameter B of Postelnicu exceeds a critical value convection commences but as oscillatory convection, not stationary convection, as previously thought.

1. Introduction

The study of thermal convection problems driven by chemical reactions is dominated by applications and has recently been witnessing much activity. For convection in a pure fluid McTaggart and Straughan [8] developed an energy method to yield global nonlinear stability thresholds and further references may be found therein. Within the field of convection in porous media the area of chemical reactions is perhaps newer and richer. McKay [7] studies the effect of chemical reaction upon convection in a porous layer overlain by a viscous fluid. Rahman and Al-Lawatia [12] investigate the effect of high order chemical reactions while Mahdy [5] continues the analysis in double diffusive convection. Andres and Cardoso [1] study instability in the presence of a chemical reaction in a finite layer and link this to the important environmental topic of carbon dioxide storage. Phase change effects may also play a prominent role, for example in compositional convection where material is deposited at the Earth's solid inner core, see e.g. Eltayeb et al. [3,4]. Nguyen et al. [9] study flow driven by a chemical reaction on an anisotropic porous cylinder, Malashetty and Biradar [6] analyse the onset of double diffusive convection with a chemical reaction present in an anisotropc porous layer. Of particular interest to the present article is the paper by Postelnicu [11] who models the situation where a chemical reaction at the base of a horizontal layer of porous material gives rise to a convective instability.

Our major goal here is to revisit the situation studied by Postelnicu [11] where we have a layer of Darcy porous material saturated with a non-isothermal, incompressible viscous fluid, in which is dissolved a chemical of concentration C. As there are several misprints in Postelnicu [11] we rederive the basic equations at the outset.

Firstly we assume Darcy's Law and then

$$\nabla p = -\frac{\mu}{K}\mathbf{v} - \rho g \mathbf{k}, \tag{1}$$

where $\mathbf{v}$, p are velocity and pressure, ρ, g are the density and gravity (acting in the negative z-direction), $\mathbf{k} = (0,0,1)$, μ is the dynamic viscosity of the fluid and K is the permeability of the porous medium. The Boussinesq approximation is adopted whereby the density is assumed constant everywhere except in the body force term in (1). In line with Postelnicu [11], the density in (1) is taken to depend only on the temperature, T, and not on the concentration C. This dependence is assumed linear so that

$$\rho = \rho_0(1 - \alpha(T - T_0)), \tag{2}$$

ρ_0 being the density at reference temperature T_0, and α is the coefficient of thermal expansion. Insertion of (2) in (1) leads to the momentum equation

$$p_{,i} = -\frac{\mu}{K} v_i - \rho_0 g (1 - \alpha(T - T_0)) k_i, \tag{3}$$

where standard indicial notation is employed.

The fluid is incompressible so that

$$v_{i,i} = 0. \tag{4}$$

The temperature field and the concentration field satisfy the equations, see Straughan [13],

$$\begin{aligned} &\frac{1}{M} T_{,t} + v_i T_{,i} = \kappa \Delta T, \\ &\phi C_{,t} + v_i C_{,i} = \phi k_c \Delta C, \end{aligned} \tag{5}$$

where $M = (\rho_0 c_p)_f / (\rho_0 c)_m$ with $(\rho_0 c)_m = \phi(\rho_0 c_p)_f + (1 - \phi)(\rho c)_s$ and $\kappa = k_m / (\rho_0 c_p)_f$ is the thermal diffusivity of the porous medium, k_m

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being given by $k_m = \kappa_s(1-\phi) + \kappa_f\phi$. The terms c_p and c_s are the specific heat at constant pressure of the fluid and the specific heat of the solid, respectively, and ϕ is the porosity of the solid.

The saturated porous medium occupies the infinite layer $\{(x,y)\in\mathbb{R}^2\}\times\{z\in(0,h)\}$ and on the upper wall the temperature and reactant concentration are fixed and constant with zero mass flux there. Thus the upper boundary conditions are

$$T = T_U, \quad C = C_U, \quad v_i n_i = w = 0, \quad \text{on } z = h. \tag{6}$$

On the lower wall ($z = 0$) there is an exothermic reaction where the reactant is converted into an inert product at a rate, r, where

$$r = k_0 C \exp\left(-\frac{E}{R^* T}\right), \tag{7}$$

k_0 is a rate constant, R^* is the universal gas constant and E is the activation energy of the reaction. The rate of change of temperature is proportional to the heat of the reaction, Q, the rate at which it occurs, and inversely proportional to the rate at which the heat is conducted away from the surface, k_T. The rate of change of the reactant concentration is proportional to the rate of the reaction and inversely proportional to the porosity of the solid and the diffusivity of the reactant, k_c. There is zero mass flux across the lower wall, hence the boundary lower boundary conditions are

$$\begin{aligned} &k_T\frac{\partial T}{\partial z} = -Qk_0 C\exp\left(-\frac{E}{R^*T}\right),\\ &\phi k_c\frac{\partial C}{\partial z} = k_0 C\exp\left(-\frac{E}{R^*T}\right),\\ &v_i n_i = w = 0 \quad \text{on} \quad z = 0. \end{aligned} \tag{8}$$

2. Basic solution and perturbation equations

At the steady state, $(\bar{\mathbf{v}}, \overline{T}, \overline{C}, \bar{p})$, the fluid velocity is zero and the temperature and reactant concentration are independent of time; $\bar{v}_i = 0$, $\overline{T}_{,t} = 0$, $\overline{C}_{,t} = 0$. If we now consider the temperature and reactant concentration fields to be functions of z only (5) immediately imply that

$$\begin{aligned} &\overline{T}(z) = \beta_1 z + \beta_2,\\ &\overline{C}(z) = \beta_3 z + \beta_4. \end{aligned} \tag{9}$$

The steady pressure, $\bar{p}(z)$, is found from (3).

We now introduce small perturbations u_i, θ, γ, π to the steady state such that

$$\begin{aligned} &v_i = \bar{v}_i + u_i, \quad T = \overline{T} + \theta,\\ &C = \overline{C} + \gamma, \quad\ \ p = \bar{p} + \pi. \end{aligned}$$

Eqs. (3)–(5) then become

$$\begin{aligned} &\frac{1}{M}\theta_{,t} + \beta_1 w + u_i\theta_{,i} = \kappa\Delta\theta,\\ &\phi\gamma_{,t} + \beta_3 w + u_i\gamma_{,i} = \phi k_c\Delta\gamma,\\ &u_{i,i} = 0,\\ &\pi_{,i} = -\frac{\mu}{K}u_i + \rho_0 g\alpha\theta k_i, \end{aligned} \tag{10}$$

where $w = u_3$.

Eq. (10) are linearised by considering the terms $u_i\theta_{,i}$ and $u_i\gamma_{,i}$ to be small and we may therefore neglect them. The pressure term is removed by taking *curl curl* of $(10)_4$ and retaining the third component ($i = 3$). The non-dimensionalisations

$$\begin{aligned} &x_i = hx_i^*, \quad\ t = \tfrac{h^2}{\kappa M}t^*,\\ &v_i = \tfrac{\kappa}{h}v_i^*, \quad C = C_U C^*,\\ &T = T_U T^*, \quad p = \tfrac{\kappa\mu}{K}p^* \end{aligned} \tag{11}$$

are employed to find

$$\begin{aligned} &\theta_{,t} + \beta_1 w = \Delta\theta,\\ &M\phi\gamma_{,t} + \beta_3 w = \frac{1}{Le}\Delta\gamma,\\ &\Delta w - R\Delta^*\theta = 0, \end{aligned} \tag{12}$$

where $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian, $Le = \kappa/(\phi k_c)$ is the Lewis number and the Rayleigh number, R, is defined by $R = Khg\rho_0\alpha T_U/(\kappa\mu)$.

By standard linear analysis we wish to find time dependent horizontally periodic solutions, therefore w, θ and ϕ take the Fourier form

$$\begin{aligned} &w = \sum_{j=1}^{\infty} e^{\sigma_j t} f_j(x,y) W_j(z),\\ &\theta = \sum_{j=1}^{\infty} e^{\sigma_j t} f_j(x,y)\Theta_j(z),\\ &\gamma = \sum_{j=1}^{\infty} e^{\sigma_j t} f_j(x,y)\Phi_j(z), \end{aligned} \tag{13}$$

where f is some function such that $\Delta^* f = -k^2 f$ for a wave number k, and σ_j is an eigenvalue (growth rate).

Since we are interested in the first instance of instability, we choose one mode of the Fourier expansions to find a typical eigenvalue which satisfies the system of equations

$$\begin{aligned} &\sigma\Theta + \beta_1 W = (D^2 - k^2)\Theta,\\ &M\phi\sigma\Phi + \beta_3 W = \frac{1}{Le}(D^2 - k^2)\Phi,\\ &(D^2 - k^2)W + Rk^2\Theta = 0, \end{aligned} \tag{14}$$

where $D = d/dz$.

3. Perturbation boundary conditions

After non-dimensionalising using (11) the upper boundary conditions are

$$w = 0, \quad C = 1, \quad T = 1 \quad \text{on } z = 1 \tag{15}$$

and the lower boundary conditions (on $z = 0$) are

$$\begin{aligned} &\frac{\partial T}{\partial z} = -AC\exp\left(\frac{-\xi}{T}\right),\\ &\frac{\partial C}{\partial z} = BC\exp\left(\frac{-\xi}{T}\right),\\ &w = 0, \end{aligned} \tag{16}$$

where

$$A = \frac{Qhk_0 C_u}{k_T T_U}, \quad B = \frac{hk_0}{\phi k_c}, \quad \xi = \frac{E}{R^* T_U}$$

are non-dimensional coefficients.

Eqs. (15), (16) and (19) imply that the linearised perturbation boundary conditions are

$$W = \Theta = \Phi = 0, \quad \text{on } z = 1 \tag{17}$$

and

$$W = 0, \quad \frac{d\Theta}{dz} = -A\Phi, \quad \frac{d\Phi}{dz} = B\Phi \quad \text{on } z = 0, \tag{18}$$

where w is now the third component of $\mathbf{u}$.

Evaluating the steady state, (9), on the boundary walls one obtains the relations