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Linear stability and adjoint sensitivity analysis of thermoacoustic networks with premixed flames

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ABSTRACT

We analyse the linear response of laminar conical premixed flames modelled with the linearised front-track kinematic *G*-equation. We start by considering the case in which the flame speed is fixed, and travelling wave velocity perturbations are advected at a speed different from the mean flow velocity. A previous study of this case contains a small error in the Flame Transfer Function (FTF), which we correct. We then allow the flame speed to depend on curvature. No analytical solutions for the FTF exist for this case so the FTF has to be calculated numerically as its parameters – aspect ratio, convection speed and Markstein length – are varied. Then we consider the stability and sensitivity of thermoacoustic systems containing these flames. Traditionally, the stability of a thermoacoustic system is found by embedding the FTF within an acoustic network model. This can be expensive, however, because the FTF must be re-calculated whenever a flame parameter is varied. Instead, we couple the linearised *G*-equation directly with an acoustic network model, creating a linear eigenvalue problem without explicit knowledge of the FTF. This provides a simple and quick way to analyse the stability of thermoacoustic networks. It also allows us to use adjoint sensitivity analysis to examine, at little extra cost, how the system's stability is affected by every parameter of the system.

1. Introduction

Thermoacoustic oscillations are one of the most persistent problems faced by rocket and aircraft engine manufacturers. They occur when heat release fluctuations lock into acoustic pressure oscillations inside a combustion chamber [1,2]. The manufacturer's ultimate aim is to design an engine that is linearly stable to thermoacoustic oscillations over the entire operating range. This is currently achieved by extensive experimental testing, repeated re-design, and sometimes by the retro-fitting of damping devices such as Helmholtz resonators. There is therefore considerable industrial motivation to develop analytical and numerical tools that can predict whether thermoacoustic oscillations will occur in a system and, if so, how to change the system in order to damp them. This requires a reliable linear model of the thermoacoustic system and is aided by the application of adjoint sensitivity analysis, as described in this paper.

The stability of a thermoacoustic system is usually analysed by first calculating the Flame Transfer Function (FTF). This is the flame's heat release response to velocity, pressure, or equivalence ratio perturbations. The FTF is then combined with an acoustic network model. In this study we will focus on the response of laminar, conical flames, modelled with the linearised *G*-equation. Previous studies have shown that a kinematic description of the flame front, using a front-tracking version of the *G*-equation with a suitable velocity model, can capture the main features of the heat release response of conical premixed flames to inlet velocity fluctuations. Birbaud et al. [3] have shown that acoustic perturbations are responsible for the formation of velocity perturbations that are advected along the flame at a characteristic speed, which in general is a function of the amplitude and frequency of the forcing oscillation. A travelling wave model of axial velocity perturbations captures this phenomenon, and radial velocity fluctuations are found by choosing a divergence free flow field. In some studies in the literature the radial component was neglected, because it was shown to be less important than the axial travelling wave on the flame response [4]. The *G*-equation moving into a travelling wave velocity field has been shown to capture some features of conical flames dynamics – such as the formation of wrinkles on the flame surface – and the consequent heat release response.

This model has been developed and compared with experiments in several studies: [5] considered an axial dependence of the mean flow field; [6] compared the responses of conical and *V*-shaped flames; [7] compared FTFs from experiment with analytical results from [4,6]; [8,9] investigated the effect of confinement on conical flames FTFs and compared with *G*-equation based analytical models;

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[10] extracted a frequency-dependent convection speed from DNS and used it into a *G*-equation low-order model. For a complete review of premixed combustion and acoustic waves coupling see [11]. In this study we will assume that perturbations travel with a constant speed, which in general is different from the mean flow velocity. We also allow the flame speed to vary linearly with the local flame curvature, which avoids the formation of unphysical cusps on the flame surface. This has already been considered for *V*-shaped flames modelled with the *G*-equation [12,13], but not for conical flames, which is a typical experimental configuration [14,15]. The flame model we derive only captures some of the characteristics of conical flames dynamics. We do not model other effects which may be important such as gas expansion [16] and flame base motion [17] to keep the low-order flame model simple.

The analysis of FTFs helps to explain the linear dynamics of flames, such as a conical flame's low-pass characteristics. Analytical results are usually not available, however, meaning that simulations or experiments over a large range of frequencies are required, which can be expensive. This becomes even more demanding when one wants to investigate the effect of several parameters on the stability of a thermoacoustic system, because a new FTF has to be evaluated for every set of parameters. On the other hand, if a relation between the flame's heat release response and acoustic velocity or pressure fluctuations is known, one can apply classic linear stability techniques to the fully-coupled system, avoiding the explicit evaluation of flame and acoustic transfer functions [18].

The aim of this paper is to apply linear stability and adjoint sensitivity techniques to a thermoacoustic network in which the flame and heat release dynamics are modelled by the kinematic *G*-equation. With this approach, the problem of identifying thermoacoustic instabilities is reduced to a classic eigenvalue problem of the coupled system. The resulting eigenvalue problem is small and all the eigenvalues can be calculated as the parameters of the model change. If the model were larger, a similar approach could be used, but only the eigenvalues with the largest growth rates would be calculated, using iterative methods.

Having obtained a description of the linear coupled thermoacoustic system, adjoint sensitivity analysis can be applied [19–22]. In this study we apply it to systems with premixed flames, modelled with the front-track *G*-equation.[1] Sensitivity results can be used, for example, to calculate how to change the system in order to reduce the growth rate of each unstable eigenvalue. This change could be in the shape of the combustion chamber, the shape of the flame, or the acoustic boundary conditions. In this paper, we demonstrate the usefulness of adjoint methods by calculating how the convection speed affects the most unstable eigenvalues.

The paper is structured as follows: in Section 2 the linear flame model is derived. In Section 3 we solve the equations for the case where the flame speed is uniform and we present analytical results that correct the FTF expression contained in [4]. In Section 3.1 we extend the model to the more general case of curvature-dependent flame speeds. In Section 4 we present the fully-coupled thermoacoustic system. In Section 4.2 we calculate its eigenvalues while varying two parameters, build a stability map and discuss the results. Finally, in Section 5, we perform a sensitivity analysis on the convection speed on the entire stability map calculated with stability analysis, and provide physical insights based on these results. In Section 6 we summarise our study and discuss the potential applications of these methods to larger problems.

2. General framework

We describe the premixed flame's dynamics with the kinematic *G*-equation model, assuming that there is no density jump across the flame. This assumption precludes the Darrieus–Landau instability in the flame. This instability can cause the formation of small-scale wrinkles leading to turbulence [16,23,24]. This, and other physical phenomenon such as reaction mechanisms and turbulence effects, can be taken into account in an LES simulation with a *G*-equation formulation, see for example [25]. However, for our purposes we want to keep the model low-order, and we consider a laminar flame, assuming that the flame is an infinitely thin interface separating reactant and products and neglecting temperature variations across the flame. Under this assumption the *G*-equation model reads:

$$\frac{\partial \tilde{G}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{G} = \tilde{s}_L^0(1 - \mathcal{L}\tilde{\kappa})|\tilde{\nabla}\tilde{G}|, \tag{1}$$

where $\tilde{\boldsymbol{u}}$ is a prescribed flow field, $\tilde{s}_L^0$ is the propagation speed of a laminar flat flame, $\mathcal{L}$ is the Markstein length, and $\tilde{k}$ is the local flame curvature. The flame front is identified by the $\tilde{G} = 0$ level set. We describe axisymmetric flames in the laboratory framework, indicating with $\tilde{r}$ and $\tilde{x}$ the radial and axial directions, respectively. We denote mean quantities with overlines and perturbations with primes. Dimensional quantities are indicated with a tilde. We also assume that the mean flow is uniform in the axial direction, and that the axial flow perturbations do not depend on the radial component. Radial velocity fluctuations are found by solving the continuity equation, assuming that the flow is incompressible. This is a well-established model that has been shown to accurately reproduce experimentally determined conical FTFs when coupled with the *G*-equation dynamics. Comparisons between FTFs determined from experiments and *G*-equation models that use this type of flow field can be found in [5,8,10]. Therefore we can write the two components of $\tilde{\boldsymbol{u}}$ as:

$$\tilde{u}_x = \overline{U}(1 + \epsilon u_x'(x, t)) \qquad \tilde{u}_r = -\frac{1}{2}\overline{U}\tilde{r}\epsilon\frac{\partial u_x'}{\partial \tilde{x}} \tag{2}$$

where $\epsilon \ll 1$ is the perturbation parameter, and $u_x' \sim \mathcal{O}(1)$ is the axial fluctuation, which can be forced or self-excited. We will consider forced fluctuations in order to examine how FTFs are affected by changes in flame speed due to curvature. However, we will not use these FTF results in the self-excited configuration. Instead, we will rewrite the equations in the frequency domain so that linear stability and adjoint methods can be applied without an explicit knowledge of the FTF.

Because we study small perturbations and are interested in the linear limit, it is correct to assume that the flame front is single-valued in a well-chosen reference system. The linearised conical flame front is always single valued in the laboratory framework with respect to the radial coordinate, with $\tilde{r}$ spanning the range $[0, R]$ at any instant.[2] Thus the $\tilde{G} = 0$ level set is expressed as:

$$\tilde{G}(\tilde{x}, \tilde{r}, \tilde{t}) = \tilde{x} - \tilde{\overline{F}}(\tilde{r}) - \epsilon\tilde{f}(\tilde{r}, \tilde{t}) = 0 \tag{3}$$

where $\tilde{\overline{F}}$ and $\tilde{f}$ are the explicit functions that define the shape of the mean flame and its perturbation, respectively.

For an axisymmetric surface $\tilde{x} = S(\tilde{r})$, the mean curvature is expressed by:

$$\tilde{\kappa}(S) = \frac{\frac{d^2S}{d\tilde{r}^2}}{\left(1 + \left(\frac{dS}{d\tilde{r}}\right)^2\right)^{3/2}} + \frac{dS}{dr}\frac{1}{r\sqrt{1 + \left(\frac{dS}{dr}\right)^2}} \tag{4}$$

[1] Note that it would be troublesome to use this approach on the entire *G*-field, because the *G*-field itself has no physical meaning away from the $G = 0$ level set, and care must be taken in calculating the sensitivities. This problem does not appear with a linearised formulation, because only the flame front position is tracked, which is a physical quantity.

[2] The conical flame is single valued also in the axial direction. However, to work with a function which has $\tilde{x}$ as an independent variable is an unfortunate choice, because the flame tip moves along this direction and the domain of existence of the flame front becomes time dependent, $[0, \tilde{x}_{\text{end}}(\tilde{t})]$, unnecessarily complicating the formulation [26].

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