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Identification of spacewise and time dependent source terms in 1D heat conduction equation from temperature measurement at a final time

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ABSTRACT

Inverse problems of identifying the unknown spacewise and time dependent heat sources $F(x)$ and $H(t)$ of the variable coefficient heat conduction equation $u_t = (k(x)u_x)_x + F(x)H(t)$ from supplementary temperature measurement $(u_T(x){:=}u(x,T_f))$ at a given single instant of time $T_f > 0$, are investigated. For both inverse source problems, defined to be as ISPF and ISPH respectively, explicit formulas for the Fréchet gradients of corresponding cost functionals are derived. Fourier analysis of these problems shows that although ISPF has a unique solution, ISPH may not have a unique solution. The conjugate gradient method (CGM) with the explicit gradient formula for the cost functional $J_1(F)$ is then applied for numerical solution of ISPF. New collocation algorithm, based on the piecewise linear approximation of the unknown source $H(t)$, is proposed for the numerical solution of the integral equation corresponding to ISPH. The proposed two numerical algorithms are examined through numerical examples for reconstruction of continuous and discontinuous heat sources $F(x)$ and $H(t)$. Computational results, with noise free and noisy data, show efficiency and high accuracy of the proposed algorithms.

1. Introduction

We study the following inverse source problems of determining the unknown source terms $F(x)$ or $H(t)$, in the following heat conduction problem:

$$\begin{cases} u_t = (k(x)u_x)_x + F(x)H(t), & (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), & x \in (0,l), \\ u(0,t) = 0, \quad u(l,t) = 0, & t \in (0,T_f], \end{cases} \tag{1}$$

from the supplementary temperature measurement:

$$u_T(x) := u(x,T_f), \quad x \in (0,l), \tag{2}$$

at the final time $T_f > 0$. Here $\Omega_T := \{(x,t) \in \mathbb{R}^2 : x \in (0,l), t \in (0,T_f]\}$. The function $u_0(x)$ (initial temperature) and the final data $u_T(x)$ satisfy the consistency conditions $u_0(0) = u_0(l) = 0$ and $u_T(0) = u_T(1) = 0$, respectively. Note that the measured data $u_T(x)$ may have a noise.

For a given source terms $F(x)$ and $H(t)$ the problem (1) is defined to be the *direct problem*. When the function $H(t)$ is known and $F(x)$ needs to be defined, the *problem of identifying the unknown spacewise heat source* $F(x)$ in (1) and (2) is defined to be as ISPF. In the case when the function $F(x)$ is given and $H(t)$ needs to be defined, the problem of identifying the unknown time dependent heat source $H(t)$ in (1) and (2) is defined to be as ISPH. The temperature distribution $u_T(x)$ given at the final time $T_f > 0$ is defined to be the *measured output data*.

Heat source identification problems are the most commonly encountered inverse problems in heat conduction. These problems have been studied over several decades due to their significance in a variety of scientific and engineering applications [1–17,19]. In many heat conduction and diffusion problems, the source terms are unknown and usually are not easy to detect directly. Instead, one of the following typical measured output data are available and feasible from the viewpoint of the experiments:

$$\begin{cases} u_T(x) := u(x,T_f), & x \in (0,l) \text{ (measured final data)}; \\ f_0(t) := -k(0)u_x(0,t), & x \in (0,l) \text{ (measured left flux)}; \\ \mu_0(0,t) := u(0,t), & t \in (0,T_f] \text{(measured temperature} \\ & \quad \text{on the left end of a rod)}. \end{cases}$$

These conditions are defined to be *overspecified boundary (measured) data*, according to inverse problems terminology. Note that in the case of the measured output data $\mu_0(0,t){:=}u(0,t)$ one needs to impose the flux condition $-k(0)u_x(0,t) = f_0(t)$ in the direct problem (1), instead of the condition $u(0,t) = 0$.

For the inverse source problem (ISP) governed by Eqs. (1) and (2) there are many studies as can be seen from the papers by Cannon and Duchateau [4] for identifying the source term depending on u (i.e. $F(x)H(t) \equiv F(u)$ in Eq. (1)), Reeve and Spiak [11], Savateev and Duchateau [5] for identifying the distributed source term. Recovery of a special form source term of a parabolic

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Nomenclature

$E(n)$	accuracy error
$e(n)$	convergence error
$F(x)$	spacewise heat source
$f_0(t)$	heat flux (W/m^2)
$H(t)$	time dependent source
h	space stepsize
$J_1(F)$	cost functional for $F(x)$
$J_2(H)$	cost functional for $H(t)$
$k(x)$	thermal conductivity (W/(mK))
l	length of rod (m)
M	number of basis functions
N_{CGM}	CGM iteration number
N_t	time mesh points number
N_x	space mesh points number
$p^{(n)}$	nth descent iteration
T_f	final time
t	time variable (s)
t_j	time mesh points
$u_0(x)$	initial temperature (K)
u_h	temperature vector y_i^j
$u_T(x)$	final time temperature
x	space variable (m)
x_i	space mesh point

Greek symbols

α	CGM iteration parameter
γ	noise level
ε_F	accuracy error for F
ε_H	accuracy error for H
ε_J	stopping parameter
ε_u	computational noise level
λ_n	eigenvalues
τ	time stepsize
$\varphi(x,t)$	adjoint problem solution

Subscripts and superscripts

f	index of the final time
i	space index
J	stopping parameter index
n	nth iteration in CGM
T	final temperature index
γ	noise index

equation with variable coefficients from an overdetermination in the form of a time integral of solution have been studied by Prilepko and Tkachenko [6]. Many researchers sought the heat source as a function of only space or time. Thus ISPs for the spacewise source term have been considered by Yang et al. [2], Liu [3], and for the spacewise source term have been considered by Farcas and Lesnic [7], Johansson and Lesnic [8], and Ling et al. [9]. Numerical solution for the simultaneous identification of temperature, temperature gradient, and general source terms in the one-dimensional inverse heat conduction problem, based on the mollification method and a marching scheme have been proposed by Yi and Murio [16]. For the general type source term $F(x,t)$ the mathematical analysis of the least square approach for ISPs, based on weak solution theory for PDEs, have been proposed by Hasanov [17], and also Hasanov et al. [19]. From the viewpoint of numerical methods and algorithms for the above ISPs, all these approaches can be divided into the following groups. In the first group, the Green's function method is used to transform the ISP to an operator equation of the first kind which is known to be moderately ill-posed. Then various recursive algorithms for the ISP, with the boundary element method for solving the direct problem, are emplyed (see, for example, [7,8]). In the second group of studies, a traditional approach is used to reduce an ISP to the frst kind Volterra integral equation, and then apply some regularization techniques to solve the ill-posed problem. Following this type formulation, Maalek Ghaini [15] has proven the existence, uniqueness and stability theorems, although no numerical procedures and examples were presented. In the next more wide group of studies least square approach with subsequent use of gradient methods, in particular, Landweber type iteration algorithms for the numerical solution of the ill-posed operator equation, are used (see [11] and references therein). Finally, there are various partial identification approaches, similar to Yan et al. [14], where the ISP is transformed into a three-point boundary value problem.

This study presents a systematic analysis of inverse source problems for the distributed source term case $F(x)H(t)$, and aims to estimate as accurately as possible the spacewise ($F(x)$) and time dependent ($H(t)$) source terms, under the overspecified data $u_T(x)$ at the final time $T_f > 0$, given by (2). The analysis is based on the proposed variational approach which permits to derive explicitly gradients of the both cost functionals:

$$\begin{cases} J_1(F) = \int_0^l \left[u(x,T_f;F) - u_T(x)\right]^2 dx, \\ J_2(H) = \int_0^l \left[u(x,T_f;H) - u_T(x)\right]^2 dx, \end{cases} \tag{3}$$

corresponding to the above defined problems ISPF and ISPH. Note that different from the above sited works, here the thermal conductivity $k(x)$ is assumed to be not a constant, which permits one to analyze also composite materials. The second main contribution of the presented study is the comparative analysis of the inverse problems ISPF and ISPH. Fourier analysis of these problems shows that although ISPF has a unique solution, ISPH may not have a unique solution. The conjugate gradient method (CGM) with the derived explicit formula for the gradient of the cost functional $J_1(F)$ is then applied for numerical solution of ISPF. New collocation algorithm, based on the piecewise linear approximation of the unknown source $H(t)$, is proposed to define the degree of ill-posedness of ISPH on the one hand, and to construct new collocation algorithm for the numerical solution of integral equation corresponding to ISPH, on the other hand. Note that the CGM does not work for ISPH.

The paper is organized as follows. In Section 2 the gradient formulas for the cost functionals $J_1(F)$ and $J_2(H)$ are derived. Section 3 is devoted to the comparative Fourier analysis of the problems ISPF and ISPH. Numerical algoritm for the direct and adjoint problems, as well as numerical examples related to estimation of the computational noise level ε_u and also the stopping parameter ε_J, are discussed in Section 4. In Section 5 the numerical results for the CGM applied to the ISPF are presented for various noise free and noisy output data. Numerical algorithm for the integral equation corresponding to ISPH, and computational results related to reconstruction of the time dependent source $H(t)$ are presented in Section 6. Some concluding remarks are given in Section 7.

2. Gradient formulas for the cost functionals of the inverse source problems

Consider first the cost functional $J_1(F)$ corresponding to ISPF. Let us denote by $u = u(x,t;F)$ the solution of the heat conduction problem (1), corresponding to the given source term $F \in \mathcal{F}$, where