Computers and Chemical Engineering 94 (2016) 75–91

Contents lists available at ScienceDirect

Computers and Chemical Engineering

journal homepage: www.elsevier.com/locate/compchemeng

Comprehensive Pareto Efficiency in robust counterpart optimization

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ARTICLE INFO

Article history:
Received 30 March 2016
Received in revised form 27 June 2016
Accepted 26 July 2016
Available online 3 August 2016

Keywords:
Robust optimization
Pareto optimality
Linear programming
Integer programming

ABSTRACT

In this paper, an innovative concept named Comprehensive Pareto Efficiency is introduced in the context of robust counterpart optimization, which consists of three sub-concepts: Pareto Robust Optimality (PRO), Global Pareto Robust Optimality (GPRO) and Elite Pareto Robust Optimality (EPRO). Different algorithms are developed for computing robust solutions with respect to these three sub-concepts. As all sub-concepts are based on the Probability of Constraint Violation (PCV), formulations of PCV under different probability distributions are derived and an alternative way to calculate PCV is also presented. Numerical studies are drawn from two applications (production planning problem and orienteering problem), to demonstrate the Comprehensive Pareto Efficiency. The numerical results show that the Comprehensive Pareto Efficiency has important significance for practical applications in terms of the evaluation of the quality of robust solutions and the analysis of the difference between different robust counterparts, which provides a new perspective for robust counterpart optimization.

1. Introduction

Robust optimization (RO), originally introduced by Soyster (1973) and later revitalized by Ben-Tal and Nemirovski (2002), Bertsimas and Sim (2004), El Ghaoui et al. (1998) in late 1990s and early 2000s, is a technique for handling uncertainties in mathematical programming problems. In RO, an uncertainty set is firstly determined, then a robust counterpart (RC) of the original optimization problem is formulated where the solution should be feasible for any uncertain realizations in the uncertainty set. The objective of RO is to calculate a robust solution which satisfies a decision-maker's requirement (e.g., a robust solution with high quality objective value and reliability). For general review and comprehensive explanation on RO, we refer to Ben-Tal and Nemirovski (2002), Ben-Tal et al. (2009), Bertsimas et al. (2011), Gabrel et al. (2014), Gorissen et al. (2015).

The definition of the uncertainty set plays an important role in RO. It directly determines the underlying RC and then affects the whole process of RO. Many works have devoted to the construction of the uncertainty sets. The first one is considered by Soyster (1973) in which all possible uncertain realizations are included. This uncertainty set is too pessimistic and conservative which is not preferred in practice. Later El Ghaoui et al. (1998), Ben-Tal and Nemirovski (2002) consider ellipsoidal uncertainty sets and the resulting RC is a second-order cone programming (SOCP). Bertsimas and Sim (2004) define a budgeted uncertainty set which leads to a linear programming (LP). This uncertainty set is further improved by Ke et al. (2013) which is called proportion-based uncertainty set specifically suitable for 0–1 integer programming problems. Bertsimas et al. (2004) generalize the definition of the uncertainty sets by more general norms. In particular, the l_1 and l_∞ norms result in linear programming problems, and the l_2 norm results in a second-order cone programming problem. Li et al. (2011) presents a systematic study on different uncertainty sets defined by different norms and their combinations for linear and mixed integer programming problems and derived corresponding RC. Other works related to uncertainty set construction include Bertsimas and Brown (2009) which construct the uncertainty set from coherent risk measures perspective, Ben-Tal et al. (2013), Bertsimas et al. (2013) construct the uncertainty set from a data-driven and statistics perspective, etc.

With the uncertainty set defined, the robust optimal solution can be obtained by solving the corresponding RC. One important procedure is to check the quality of the robust solution, in order to make the right decision. One criterion of the solution quality is the objective value. When the uncertainty lies in the constraint, then the Probability of Constraint Violation (PCV) naturally becomes another criterion of the solution quality. For a decision maker, a solution with better objective value and lower PCV is always preferred. Many works have devoted to establishing the Probability Bounds of Constraint Violation (PBCV) when the distribution

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http://dx.doi.org/10.1016/j.compchemeng.2016.07.022
0098-1354/

information is unknown or partially known, we refer to Bertsimas and Sim (2004), Lin et al. (2004), Paschalidis and Kang (2015), Ben-Tal et al. (2009), Li et al. (2012), Guzman et al. (2016). To the best of our knowledge, there is no systemic work on establishing PCV with known probability distributions in RO.

Iancu and Trichakis (2013) first introduced the concept of Pareto efficiency in the context of the RO methodology for linear optimization problems. The traditional RO optimizes the objective by satisfying the uncertain constraints under all possible uncertain realizations. However, as pointed out by Iancu and Trichakis (2013), the RO does not optimize the slacks of constraints, in fact, it fails to guarantee that no other solution exists yielding larger slacks and at the same objective value. Iancu and Trichakis (2013) defines the concept of Pareto Robust Optimality (PRO) based on constraint slacks. For a decision maker, the PRO solutions are always preferred to the non-PRO solutions as the non-PRO solutions can more readily generate infeasibility. One problem exists in Iancu and Trichakis (2013) is that comparing solutions feasibility by constraint slacks is very intuitive and sometimes not so accurate. A solution with less constraint slacks may has higher feasibility. Instead the PCV is the most accurate measurement of solution feasibility. Based on this observation, we can redefine the PRO by using PCV rather than constraint slacks to improve the accuracy. The premise is that the probability distribution information is known beforehand.

In this paper, we introduce an innovative concept named Comprehensive Pareto Efficiency in the context of robust counterpart optimization for linear and 0–1 integer programming problems with uncertain constraints. The main contributions are as follows:

1. Comprehensive Pareto Efficiency is initially introduced which consists of three sub-concepts: Pareto Robust Optimality (PRO), Global Pareto Robust Optimality (GPRO) and Elite Pareto Robust Optimality (EPRO).
2. Different algorithms are developed for computing robust solutions with respect to PRO, GPRO and EPRO.
3. Formulations of PCV under different probability distributions are derived, and an alternative way for calculating PCV is also presented.
4. We draw numerical studies on two applications (production planning problem and orienteering problem), to demonstrate the Comprehensive Pareto Efficiency in terms of the evaluation of the quality of robust solutions and the analysis of the difference between different robust counterparts.

The remainder of the paper is organized as follows: Section 2 reviews the robust counterpart optimization methodology, Section 3 introduces the Comprehensive Pareto Efficiency concept which consists of three sub-concepts, Section 4 describes the calculation of PCV under different probability distributions, Numerical studies are drawn in Section 5 with two applications and Section 6 concludes the whole paper.

2. Robust counterpart optimization

In this paper, we consider the following linear programming problem and 0–1 integer programming problem simultaneously:

$$\text{LP: } \max\left\{\mathbf{c}^{\mathrm{T}}\mathbf{x} : \mathbf{A}\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0}\right\} \tag{1a}$$

$$\text{0-1 IP: } \max\left\{\mathbf{c}^{\mathrm{T}}\mathbf{x} : \mathbf{A}\mathbf{x} \le \mathbf{b}, \mathbf{x} \in \{0,1\}^N\right\} \tag{1b}$$

where $\mathbf{c} \in \mathbb{R}^N, \mathbf{A} \in \mathbb{R}^{M\times N}, \mathbf{b} \in \mathbb{R}^M$.

We only consider single uncertain constraint in this paper. Suppose the ith row of $\mathbf{A}$ is affected by uncertainty, denote the transpose of ith row of $\mathbf{A}$ as uncertain vector $\mathrm{a}_i \in \mathbb{R}^N$, each element in a_i is modeled as independent and symmetric random variable. Then the ith constraint of the nominal linear programming problem and 0–1 integer programming problem turns into

$$\mathrm{a}_i^{\mathrm{T}}\mathbf{x} \le b_i \tag{2}$$

The robust optimization methodology is thus to define a so-called uncertainty set $\mathcal{U}$ for uncertain vector a_i such that the ith constraint satisfied as:

$$\mathrm{a}_i^{\mathrm{T}}\mathbf{x} \le b_i, \quad \forall \mathrm{a}_i \in \mathcal{U} \tag{3}$$

which is known as the robust counterpart of the uncertain constraint (2).

Without loss of generality, the uncertainty set $\mathcal{U}$ is defined as follows:

$$\mathcal{U} = \{\mathrm{a}_i = \mathbf{a}_i + \mathrm{A}_i\zeta | \zeta \in \mathcal{Z}\} \tag{4}$$

where $\mathbf{a}_i$ is the nominal value, $\mathrm{A}_i = diag(\hat{\mathrm{a}}_i)$ is the perturbation set where $\hat{\mathrm{a}}_i \in \mathbb{R}^{N+}$ is the perturbation vector, $\zeta \in \mathbb{R}^N$ is the vector of primitive uncertainties, and $\mathcal{Z}$ is a convex set which can be defined by a general norm of ζ as follows:

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^N | \left\|\zeta\right\| \le \Delta\} \tag{5}$$

where $\|\cdot\|$ is any norm and Δ is the parameter controlling the size of $\mathcal{Z}$.

The key of RO is the definition of the set $\mathcal{Z}$, a particular $\mathcal{Z}$ directly determines the corresponding robust counterpart. One concern in RO is the tractability of the robust counterpart. The norm defined by ℓ_∞, ℓ_1 and ℓ_2 will lead to tractable robust counterparts (Bertsimas et al., 2004). The uncertainty set defined by ℓ_∞ is called box uncertainty set:

$$\mathcal{Z}_\infty = \{\zeta \in \mathbb{R}^N | \left\|\zeta\right\|_\infty \le \Theta\} \tag{6}$$

then the linear programming problem and 0–1 integer programming problem have the same robust counterpart:

$$RC_\infty = \max\left\{\mathbf{c}^{\mathrm{T}}\mathbf{x} : \ \mathbf{a}_i^{\mathrm{T}}\mathbf{x} + \Theta\hat{\mathrm{a}}_i^{\mathrm{T}}\mathbf{x} \le b_i\right\} \tag{7}$$

In RC_∞ only the ith constraint is presented and other constraints are eliminated to keep it concise. In the following robust counterparts we only present the ith constraint as a convention, and a robust counterpart represents a linear and an 0–1 integer programming problem simultaneously.

The uncertainty set defined by ℓ_1 is called polyhedral uncertainty set:

$$\mathcal{Z}_1 = \{\zeta \in \mathbb{R}^N | \left\|\zeta\right\|_1 \le \Gamma\} \tag{8}$$

and the corresponding robust counterpart is:

$$RC_1 = \max\left\{\mathbf{c}^{\mathrm{T}}\mathbf{x} : \begin{array}{l} \mathbf{a}_i^{\mathrm{T}}\mathbf{x} + z\Gamma \le b_i \\ z \ge \hat{a}_{ij}x_j, \quad \forall j \end{array}\right\} \tag{9}$$

The uncertainty set defined by ℓ_2 is called ellipsoidal uncertainty set:

$$\mathcal{Z}_2 = \{\zeta \in \mathbb{R}^N | \left\|\zeta\right\|_2 \le \Omega\} \tag{10}$$

and the corresponding robust counterpart is:

$$RC_2 = \max\left\{\mathbf{c}^{\mathrm{T}}\mathbf{x} : \mathbf{a}_i^{\mathrm{T}}\mathbf{x} + \Omega\sqrt{\mathbf{x}^{\mathrm{T}}\mathrm{A}_i^2\mathbf{x}} \le b_i\right\} \tag{11}$$

The above three uncertainty sets are applicable when random vector ζ is unbounded, if the random vector ζ is bounded in an interval, specifically consider $\zeta \in [-1, 1]^N$, the above three uncertainty sets need to be bounded in order to limit ζ in its bound. This leads to two more uncertainty sets which are applicable when $\zeta \in [-1, 1]^N$, the first one is the intersection of the box and polyhedral sets:

$$\mathcal{Z}_{1\cap\infty} = \{\zeta \in \mathbb{R}^N | \left\|\zeta\right\|_1 \le \Gamma, \left\|\zeta\right\|_\infty \le 1\} \tag{12}$$