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A highly accurate LGSM for severely ill-posed BHCP under a large noise on the final time data

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ABSTRACT

We consider a backward heat conduction problem (BHCP) in a slab, by subjecting to data at a final time, and some different type boundary conditions at two ends of the slab. The BHCP is known to be highly ill-posed. In order to numerically solve the BHCP we develop a new Lie-group shooting method (LGSM) in the spatial direction. It can retrieve very well the initial data with a high order accuracy. Several numerical examples of the BHCP demonstrate that the LGSM is applicable, even for those of strongly ill-posed ones with a large value of final time. Under the noisy final data the LGSM is robust against the disturbance. The new method is applicable for a severely ill-posed case with a final data very small in the order of 10^{-43}, and the noise level is in the order of 10^{-1}, of which the numerical solution still has an accuracy in the order of 10^{-2}. The results are rather significant in the computations of BHCP.

1. Introduction

The problems that appear in engineering applications can be categorized as direct problems and inverse problems. For the direct problems which are governed by ordinary differential equations (ODEs), Liu [1] has derived a Lie group transformation for the augmented dynamics on the future cone, and developed group preserving scheme (GPS) for an effective numerical calculation of the nonlinear ODEs. Recently, the GPS is also proved to be very effective to deal with the ODEs equipped with special structures, as shown by Liu [2] for stiff equations, and by Liu [3] for ODEs with multiple constraints.

Numerical schemes adopted for the backward problems are usually implicit. The explicit ones that have been applied to solving the backward problems are apparently not very effective up to now. As mentioned by Mera [4], the backward problem is impossible to be solved by using the classical numerical methods and requires special techniques.

In this paper we develop a new Lie-group shooting method (LGSM) for the backward heat conduction problem (BHCP). It is an extension of the works by Liu [5], Liu et al. [6] and Chang et al. [7]. Chang et al. [8] have extended the new shooting method in [7] to solve the multi-dimensional BHCPs. Recently, Liu [9–11] and Yeih and Liu [12] have explored its superiority by using the LGSM to estimate parameters in the parabolic type PDEs, as well as to identify the heat source. The BHCP is one of the inverse problems for the applications in the heat conduction engineering to recovering the past temperature history. The inverse problems are those in which one would like to determine the causes for a desired or observed effect. One of the characterizing properties of many of the inverse problems is that they are usually ill-posed. Mathematically speaking, the linear operator generated from the BHCP is a compact one with infinite rank, whose inverse is discontinuous, and thus, the solution that depends continuously on the final data does not exist. The new LGSM would render a more compendious numerical implementation than other schemes to solve the BHCP. The degree of the ill-posedness of BHCP is over other inverse heat conduction problems, including the sideways heat conduction problem [13].

We consider the following BHCP:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < t < T, \quad 0 < x < \ell, \tag{1}$$

$$u(0,t) = u_0(t), \quad u(\ell,t) = u_\ell(t), \tag{2}$$

$$u(x,T) = u^T(x), \tag{3}$$

where $u^T(x)$ is a given final datum, and $u_0^2(t) + u_\ell^2(t) > 0$. It means that the boundary conditions cannot be both zero. After developing the LGSM in Section 4, we also consider other possible boundary conditions, given by

$$u(0,t) = u_0(t), \quad v(\ell,t) = v_\ell(t), \tag{4}$$

$$v(0,t) = v_0(t), \quad u(\ell,t) = u_\ell(t), \tag{5}$$

$$v(0,t) = v_0(t), \quad v(\ell,t) = v_\ell(t), \tag{6}$$

where $v = \partial u/\partial x$ and $-v$ is the heat flux.

In order to calculate the BHCP, there appeared certain progresses in this issue, including the boundary element method

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Nomenclature

$\mathbf{A}$	augmented matrix
a, b	coefficients defined in Eqs. (28), (31), (37), (49)
$\mathbf{f}$	$2n$-dimensional vector field in Eq. (14)
$\hat{\mathbf{f}}$	$:= \mathbf{f}(\hat{x}, \hat{\mathbf{y}})$
$\mathbf{F}$	$:= \hat{\mathbf{f}}/\|\hat{\mathbf{y}}\|$
$\mathbf{F}_1$	the first n components of $\mathbf{F}$
$\mathbf{F}_2$	the last n components of $\mathbf{F}$
$F(x,t)$	transform function defined in Eq. (72)
$\mathbf{g}$	$2n+1$-dimensional Minkowski metric
$\mathbf{G}$	an element of Lorentz group
$\mathbf{G}_i$, $i = 1, \ldots, K$	elements of Lorentz group
$\mathbf{G}(r)$	an element of Lorentz group
$\mathbf{G}(\mathbf{y}^0, \mathbf{y}^\ell)$	an element of Lorentz group
G_0^0	the 00th component of $\mathbf{G}$
$\mathbf{h}$	right-hand side of Eq. (10)
$\hat{\mathbf{h}}$	$:= \mathbf{h}(r\ell, \hat{\mathbf{u}})$
h	$:= \ell/K$
$\mathbf{I}_{2n}$	$2n$-dimensional unit matrix
ℓ	length of slab
$\|\bullet\|$	Euclidean norm
$\mathbb{M}^{2n+1}$	$2n+1$-dimensional Minkowski space
n	number of discretized time points
r	weighting factor
$\mathbb{R}$	real numbers
$R(i)$	random numbers
s	level of noise
$SO_o(2n,1)$	$2n+1$-dimensional Lorentz group
$so(2n,1)$	the Lie algebra of $SO_o(2n,1)$
S	$:= \ell\|\mathbf{y}^\ell - \mathbf{y}^0\|$
t	time
T	final time
t_i	$:= (i-1)\Delta t$
Δt	time stepsize
$u(x,t)$	temperature
$u^T(x)$	final time temperature function
$u_0(t)$	left-boundary temperature function
$u_\ell(t)$	right-boundary temperature function
$u^i(x)$	$:= u(x, t_i)$
$\mathbf{u}$	temperature vector of u^i
$\mathbf{u}^\ell$	temperature vector at $x = \ell$
$\mathbf{u}^0$	temperature vector at $x = 0$
$\hat{\mathbf{u}}$	$:= r\mathbf{u}^0 + (1-r)\mathbf{u}^\ell$
$U(x,t)$	defined in Eq. (72)
$\upsilon(x,t)$	heat flux
$\upsilon_0(t)$	left-boundary heat flux
$\upsilon_\ell(t)$	right-boundary heat flux
$\upsilon^j(x)$	$:= \upsilon(x, t_i)$
$\mathbf{v}$	heat flux vector of υ^j
$\mathbf{v}^\ell$	heat flux vector at $x = \ell$
$\mathbf{v}^0$	heat flux vector at $x = 0$
$\hat{\mathbf{v}}$	$:= r\mathbf{v}^0 + (1-r)\mathbf{v}^\ell$
x	space variable
Δx	mesh size of x
$\hat{x}$	$:= r\ell$
$\mathbf{X}$	$2n$-dimensional augmented vector
$\mathbf{X}_k$	numerical value of $\mathbf{X}$ at the kth spatial step
$\mathbf{X}^0$	the value of $\mathbf{X}$ at $x = 0$
$\mathbf{X}^\ell$	the value of $\mathbf{X}$ at $x = \ell$
$\mathbf{y}$	$2n$-dimensional vector defined in Eq. (14)
$\mathbf{y}^0$	the value of $\mathbf{y}$ at $x = 0$
$\mathbf{y}^\ell$	the value of $\mathbf{y}$ at $x = \ell$
$\hat{\mathbf{y}}$	$:= r\mathbf{y}^0 + (1-r)\mathbf{y}^\ell$
Z	$:= \exp(S/\eta)$

Greek symbols

ϵ	convergence criterion
η	coefficient defined in Eqs. (34) and (46)
θ	intersection angle of $\mathbf{y}^\ell - \mathbf{y}^0$ and $\mathbf{y}^0$

Subscripts and superscripts

i	index
K	index
t	transpose

[14], the iterative boundary element method [15,16], the Tikhonov regularization technique [17,18], the operator-splitting method [19], the lattice-free high-order finite difference method [20], the contraction group technique [5], the fundamental solution method [4], the third order mixed-derivative regularization technique [21], the Fourier regularization method [22], the three-spectral regularization methods [23], and the regularization of Fredholm integral equation method [24]. The method developed by Liu [5] was further developed by Xiong et al. [25], of which the stepsize used in the spatial finite difference is deemed as a regularization parameter. Chiwiacowsky and de Campos Velho [26] have given a review of the numerical solutions of the BHCP.

Clark and Oppenheimer [27] and Ames et al. [28] have used a quasi-reversibility method to approximate the BHCP. The numerical implementation of the quasi-reversibility together with a time-direction Lie-group shooting method has been carried out by Chang et al. [7] to calculate the BHCP with a high performance.

However, when the final data are almost diminished and are polluted by a large noise, all the above methods may fail to recover the initial data. In this paper we will extend the Lie-group shooting method, which is used for the boundary value problems as originally designed by Liu [29–31] for the direct problems, to the BHCP by considering a spatial-direction Lie-group shooting technique for matching the boundary conditions. It is interesting to note that the new method does not require any a priori regularization. It also exhibits two advantages than other methods that the new method is insensitive to the final data and is insensitive to the noise disturbance. It means that the present method can be applied to retrieve the initial data for a long history of BHCP with a large time span. Through our study in this paper, it would be clear that the new method can greatly reduce the computational time and is easy to a numerical implementation for the calculation of BHCP.

2. Solution procedures

When the BHCP is considered in a spatial interval of $0 < x < \ell$ by subjecting to the boundary conditions at two ends of a slab and an available measured datum at a final time, we will solve it through two steps. In the first step we will develop a new shooting method, namely the Lie-group shooting method (LGSM), extending the works by Liu [29–31] from a two-dimensional shooting technique to a $2n$-dimensional shooting technique. Then, after searching the missing left-side boundary condition at $x = 0$ we applied the fourth order Runge–Kutta method (RK4) or the GPS to calculate the temperature distribution inside the slab in the whole past histories, including the initial one of $u(x, 0)$.

We transform Eq. (1) into the following equations:

$$\frac{\partial u(x,t)}{\partial x} = \upsilon(x,t), \tag{7}$$

$$\frac{\partial \upsilon(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t}. \tag{8}$$