International Journal of Heat and Mass Transfer 54 (2011) 3789–3795

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Improved low-order models for heat conduction problems

H. Sadat *, C. Prax

Institut PPRIME, CNRS, Université de Poitiers, ENSMA UPR 3346, Département Fluides, Thermique, Combustion, ESIP, Campus Sud 40, avenue du recteur Pineau, 86022 POITIERS Cedex, France

ARTICLE INFO

Article history:
Received 26 August 2010
Received in revised form 16 December 2010
Accepted 22 December 2010
Available online 12 April 2011

Keywords:
Heat conduction
Perturbation method
Accelerated convergence
Sinusoidal response

ABSTRACT

An explicit simple analytical method is presented for periodic heat conduction transfer in solids by using a perturbation method. Low order models are developed and their accuracy was compared to that of the complete numerical model. It is shown that first and second order models can be used efficiently for relatively low frequencies. An improvement of the method is then proposed by using a convergence acceleration of the series. This allows the use of the method at much higher frequencies.

1. Introduction

The solution of periodic heat conduction is of special interest because of its wide applications in engineering heat transfer. Finite difference [1], finite volume [2], finite element [3] or meshless [4] methods have been generally used. The usual limitation is the size of the problem. Sometimes it is necessary to use a small time step and the computing time can increase rapidly. A perturbation method has been used successfully in [5] to develop first order models which can be viewed as improved lumped models for the slab, infinite cylinder and spherical geometries in the case of a step response. A second order model has been examined in [6] for the slab and in [7,8] for the cylinder and the sphere. Furthermore, these models have been used to solve inverse heat conduction problems [9]. The aim of the present paper is to extend the ideas of [5–8] to the case of periodic heat conduction (other variable situations can be studied as well). In the next sections, the mathematical foundation and solution procedure are described first. Then, a numerical example (in two-dimensional (2-D) periodic heat conduction) is reported to illustrate the correctness of the proposed method. It is shown that the first and second order models give accurate enough results below a critical angular frequency. Finally, a convergence acceleration is used in order to extend the frequency range of validity of the proposed method. It is shown that the improved fourth order model can be used with a high accuracy for an adimensional frequency as high as 160.

* Corresponding author. Tel.: +33 5 49453810; fax: +33 5 49453545.
E-mail address: hamou.sadat@univ-poitiers.fr (H. Sadat).

0017-9310/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2010.12.042

2. Perturbation method of solution

One considers unsteady heat conduction problem in a solid of arbitrary shape, initially at a uniform temperature T_i. All the solid boundaries (Γ_1) are maintained at the constant temperature T_i excepted one (Γ_2) submitted to a variable temperature $f(t)$. In this work, we will focus on a sinusoidally varying function with an angular frequency ω.

By using the following change of variables:

$$\begin{cases} \theta = T - T_i \quad \tau = \frac{\lambda}{\rho C} t \\ \Omega = \frac{\rho C L^2}{\lambda} \omega \end{cases} \tag{1}$$

where L represents the reference length, the heat equation, the boundary and initial conditions can be written in the following adimensional form:

$$\frac{\partial \theta}{\partial \tau} = \Delta \theta \tag{2}$$

$$\begin{cases} \theta(\mathbf{x}, 0) = 0 & \text{at } \tau = 0 \\ \theta(\mathbf{x}, \tau) = 0 & \text{on } \Gamma_1 \\ \theta(\mathbf{x}, \tau) = f(\tau) & \text{on } \Gamma_2 \end{cases} \tag{3}$$

In order to develop a low order model, we introduce as in [2] the perturbation parameter ε (which will be set to one later) in the LHS of Eq. (2):

$$\varepsilon \frac{\partial \theta}{\partial \tau} = \Delta \theta \tag{4}$$

We now seek the solution in the following form:

Nomenclature

$F_n(t)$	time functions	ε	perturbation parameter
$f(t)$	forcing function	τ	dimensionless time
t	time [s]	θ	dimensionless temperature
T	temperature [°C]	ψ_n	n order spatial function
$\mathbf{x}$	vector of coordinates		
		Subscripts	
Greek symbols		i	initial
α	thermal diffusivity [m^2 s^{-1}]	ref	reference
λ	thermal conductivity [W m^{-1} K^{-1}]		

$$\theta(\mathbf{x},\tau)=\sum_{n=0}^{\infty}\varepsilon^n\psi_n(\mathbf{x})\cdot F_n(\tau) \tag{5}$$

where $\psi_n(\mathbf{x})$ is the nth order perturbation spatial function.

By introducing expression (5) into Eq. (4), and equating the different terms, one obtains the following recurrence relations:

$$\begin{cases}\Delta\psi_0=0\\ \psi_{n-1}=\Delta\psi_n \quad n>0\end{cases} \tag{6-a}$$

$$F_n(\tau)=\frac{dF_{n-1}}{dt} \quad n>0 \tag{6-b}$$

The initial and boundary conditions (1), lead to write:

$$\begin{cases}\psi_n=0 \quad \text{on } \Gamma_1\\ \psi_0=1 \quad \text{on } \Gamma_2\\ \psi_n=0 \quad \text{on } \Gamma_2 \text{ and } n>0\end{cases} \tag{7-a}$$

$$F_0(t)=f(t) \tag{7-b}$$

$$F_n(t)=\frac{d^nf(t)}{dt^n} \tag{7-c}$$

By truncating the series (5) at the first and second order, the approximate low order solutions (θ_1 and θ_2) are given by:

$$\theta_1(\mathbf{x},\tau)=\psi_0(\mathbf{x})f(\tau)+\psi_1(\mathbf{x})\frac{df}{d\tau} \tag{8-a}$$

$$\theta_2(\mathbf{x},\tau)=\psi_0(\mathbf{x})f(\tau)+\psi_1(\mathbf{x})\frac{df}{d\tau}+\psi_2(\mathbf{x})\frac{d^2f}{d\tau^2} \tag{8-b}$$

The functions ψ_0, ψ_1 and ψ_2 are obtained by solving Eqs. (6) with boundary conditions (7). The explicit solutions (8) can thus be used to calculate the temperature in the domain if the forcing function f and its derivatives are given.

3. Test problem

In this work, we consider the case of an harmonic forcing function. The desired output was the periodic temperature in the two dimensional perforated plate sketched in Fig. 1, composed of an inner hot cylinder and an outer cold unit square. The cylinder is centred at $(x, y) = (0.6, 0.6)$ and its radius is equal to 0.25. The boundary conditions are a zero temperature on the cold enclosure and a temperature varying periodically on the surface of the inner cylinder with an angular frequency ω.

This problem can be solved by using any spatial discretization method and an implicit or explicit Euler method. In this study, the diffuse approximation meshless method [10–14] is used with an implicit in time scheme. The domain is discretised with 1373 nodal points, 41 on each square wall and 63 on disc boundary (see Fig. 1). The corresponding linear system is solved at each time step. To reach a very good accuracy on the local amplitudes of the fluctuating field, once fixed the pulsation of the excitation, the time step is chosen in such a way that 160 time steps are used per period of the excitation. Fig. 2 shows for example, the amplitude of the response in the whole domain for an angular frequency $\Omega = 15$.

4. Low order solutions

In this section, we compare the results given by the complete implicit in time meshless method to those given by the approximate low order models. We have chosen to present the results for one control point situated at $(x, y) = (0.25, 0.25)$. This point is located in the region with the largest error. It is worth noting however that the same behaviour has been observed on all points of the studied domain.

To begin, let us consider the first and second order solutions which are given by Eqs. (8-a) and (8-b). The amplitudes of the solutions are sketched in Fig. 3 together with the complete numerical solution for $\Omega = 15$. It can be seen that the second order model gives accurate results while the first order model overestimates the response.

The difference between the complete numerical solution and the low order solutions has been estimated locally by using the RMS difference integrated over a period δ:

$$\delta=\int_T\sqrt{(\theta_{implicit}-\theta_{approximate})^2}d\tau \tag{9}$$

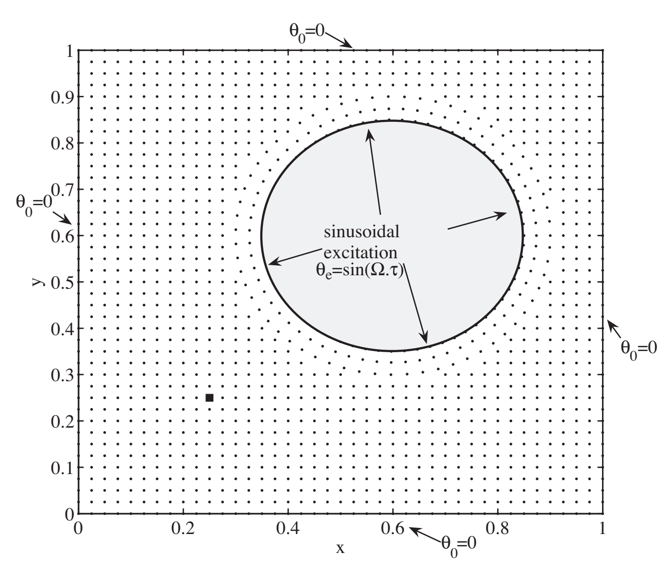

Fig. 1. Grid and boundary conditions for the simulation of the unsteady conduction in a perforated plate. The black square symbol shows the position of a control point in the grid.