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A heat-flux based "building block" approach for solving heat conduction problems

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ABSTRACT

A numerical approximation of the Green's function equation based on a heat-flux formulation is given. It is derived by assuming as a functional form of the surface heat flux a stepwise variation with space and time. The obtained approximation is very important in investigation of the inverse heat conduction problems (IHCPs) because it gives a convenient expression for the temperature in terms of the heat flux components. Additionally, it is very important for the unsteady surface element (USE) method which is a modern boundary discretization method. Green's function approximate solution equation (GFASE) also creates 'naturally' fixed groups or modules of work elements called "building blocks" that may be added together to obtain space and time values of temperature. In the current case, they are subject to a partial heating by an applied surface heat flux. The "building block" solution can be derived by using the various analytical and numerical approaches available in heat conduction literature though the exact analysis is preferable, as discussed in the text. Poorly-convergent series deriving from Green's functions approach are replaced by closed-form algebraic solutions.

1. Introduction

A transient, multi-dimensional, heat conduction problem can be solved using analytical (exact and approximate) [1,2] and numerical methods [3,4].

Two types of exact analytical procedures are available in literature. One is based on a differential formulation of the mathematical model and includes the traditional methods of separation of variables (SOV) and Laplace transforms [1]. The other is based on an integral form of the model and uses Duhamel's theorem [1,2] and Green's functions [1,2] both of which yield convolution-type integrals. These two types of exact solutions often contain infinite series, special functions, transcendental equations for eigenvalues, etc., so that their numerical computation may present a formidable task. Another exact analysis which derives its basis from the classical SOV method is the integral-transform technique [1].

Approximate analytical solutions [1] such as integral method, Galerkin method, and so on, are useful when the exact analysis is not applicable. For example, when the geometries of the heat-conducting bodies are complicated and/or the partial differential equation and the boundary conditions are nonlinear, that is, functions of temperature. In this case, in fact, only a very few special cases can be solved exactly.

Numerical methods [3,4] are useful for solving problems involving nonlinearities, complex geometries, complicated boundary conditions such as radiation conditions or mixed discontinuous boundary value problems [5]. They treat as their basic unknowns the values of the temperature at discrete points of the domain (called the 'grid' points) providing a set of algebraic equations for these unknowns. Similarly to the exact analysis, there are two types of numerical procedures. One is based on a differential formulation of the transient heat conduction equation (linear or nonlinear) and involves finite difference (FD) and finite element (FE) methods. An alternative method is the finite control-volume formulation. The other is based on an integral form of the same equation and employs Duhamel's theorem and Green's functions (GF) both of which are only valid for linear cases. As Duhamel's theorem can be thought of as a boundary condition term of the GF equation (i.e., a special case of the general method of GF), only Green's function method is here given in a numerical approximate form. A modern technique which utilizes Duhamel's integral or GF approach for solving connected basic geometries is the so-called unsteady surface element (USE) method [2, Chapter 12]. This treatment is one of the few to base the solution on GFs. Another Green' function-based numerical method is the well-established boundary element method (BEM) [6,7].

Only a tiny fraction of the range of practical problems can be solved in an exact closed-form. Notwithstanding this, the development of exact analytic solutions is relevant for verification purposes of large numerically based codes [8–11]. These solutions,

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Nomenclature

G	Green's function (subscript designates the boundary conditions)
k	thermal conductivity
L	plate thickness
q	heat flux
t	time coordinate
T	temperature
W_0	heating region
x, y	space coordinates
X_j^n	second cross derivative of temperature in space and time, $X_j^n = \nabla_j \nabla_n \theta_j^n$

Greek symbols

α	thermal diffusivity
β_m	mth dimensionless eigenvalue, $\lambda_m L$ (m = 1,2,3,...)
Δt	time interval
Δy	space interval along y
θ_j^n	temperature at $t = n\Delta t$ due to a unit heat flux applied over the region $y \in [0, j\Delta y]$
$\nabla_j \theta_j^n$	first difference (or derivative) of the temperature θ_j^n in space (backward difference)
$\nabla_n \theta_j^n$	first difference (or derivative) of the temperature θ_j^n in time (backward difference)

Superscripts

$\sim$	dimensionless variable (defined in the text)

Subscripts

j	integer index for space (j = 1,2,...,J)
n	integer index for time (n = 1,2,...,N)
x, y	x- and y-directions

in fact, not only can provide more accurate values of temperature and heat flux components within the domain of interest but their correctness can also be checked by means of intrinsic verification methods [2,12]. Also, they provide a better insight to the physical significance of various parameters affecting a given problem than a purely numerical solution.

The various direct (analytical and numerical) approaches described briefly above are the first stage of solution procedures for solving the inverse heat conduction problems (IHCPs) [13]. Among them, the numerical approximate form of the Green's function equation based on a heat-flux formulation can be relevant in investigation of the IHC problems because it gives a convenient expression for the temperature in terms of the unknown heat flux components (Sections 2 and 3). Also, it states that the temperature or heat flux computation employs only one basic "building block" solution, which is the solution of a direct problem subject to a partial heating by a boundary condition of Neumann type (Section 4).

As the above solution is not available in the literature [1,2,14], it is derived in the paper by exact analysis using Green's functions [2] and routinely evaluated numerically as integrating part of the same analysis (Sections 5 and 6). An exact analytical procedure is used since the numerical approximation of the GF equation is based on differences (close subtractions) of the building block solutions in both space and time, as discussed in Section 4. These differences, in fact, require very accurate values of the temperature, in particular when the space and time intervals chosen for approximating the surface heat flux are very small. However, the computational time is greatly reduced by using analytical solutions [15].

Effective procedures based on insights of Morse and Feshbach [16] and Beck et al. [17] allow then the poorly-convergent series derived for the steady-state case to be replaced by closed-form algebraic solutions. The algebraic forms remove the convergence problem associated with the series solution, can prove helpful in verification of numerical solutions [8–11] and are more insightful than series solutions (Sections 6.1 and 6.2). Finally, a numerical example is given to show how the proposed procedure works (Section 7).

2. Steady-state problem

Consider a plate of thickness L along x and semi-infinite in the y-direction with temperature-independent properties. The plate is thermally insulated at $y = 0$, is kept at zero temperature at $x = L$ and is subject to a time-independent heat flux at its boundary surface $x = 0$ (nonhomogeneous boundary condition of the 2nd kind). For convenience, this heat flux is assumed as a function of one space coordinate only, that is, $q_x(0,y) = q(y)$ and is applied over a finite space interval $y \in [0, y_J]$. A schematic of this two-dimensional steady-state problem denoted by X21B(y-)0Y20B0 is given in Fig. 1a; in the notation, X and Y denote the x- and y- directions, respectively, and "(y-)" indicates an arbitrary space function along y. (See more detail in Ref. [2, Chapter 2] for the numbering system devised by Beck, et al.).

Its mathematical formulation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{1a}$$

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = q(y) \quad T(L,y) = 0, \tag{1b}$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0 \quad T(x, y \to \infty) = \text{finite}. \tag{1c}$$

One way to treat this problem is to assume a functional form of the surface heat flux variation with y (function specification method) [13]. A simple way to approximate an arbitrary $q(y)$ curve is to divide the same curve into a number of equally spaced intervals, Δy, and to substitute a uniform heat flux within each of these intervals for the real $q(y)$. This gives the stepwise profile sketched in Fig. 1b where q_j (jth heat flux component) is an approximation for $q(y)$ between $y_{j-1} = (j-1)\Delta y$ and $y_j = j\Delta y$ (with j = 1,2,...,J). The heat flux q_j may be identified with the space coordinate $y_{j-1/2} = (j - 1/2)\Delta y$, that is, $q_j = q(y_{j-1/2})$, where $y_{j-1/2}$ is the coordinate of the grid point $(j - 1/2)$.

Another type of approximation of $q(y)$ is the linear elements (piecewise-linear profile) which can represent a $q(y)$ curve more accurately than the constant elements. This accuracy is gained, however, at the expense of a more complex treatment. Other possible approximations are the use of parabolas, cubics, cubic splines, or exponentials.

However, the main case considered is the simplest one, that is, the uniform element approximation of Fig. 1b. This is consistent with the various numerical approaches described in Section 1 which are based on either differential or integral forms of the transient heat conduction model. An integral formulation employing Green's functions is here used.

Then, the solution to the linear problem here of interest may be taken as [2, p. 225]

$$T(x,y) = \frac{1}{k}\int_{y'=0}^{y_M} q(y') G_{X21Y20}(x, y, x' = 0, y') dy', \tag{2}$$