Available online at www.sciencedirect.com
ScienceDirect
International Journal of Heat and Mass Transfer 51 (2008) 4583–4588
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Technical Note

A note on methods for analysis of flow through microchannels

G. Chakraborty *

Department of Mechanical Engineering, Indian Institute of Technology, Kharagpur 721 302, India

Received 30 May 2007; received in revised form 4 October 2007
Available online 20 April 2008

Abstract

The flow problem within a straight microchannel of arbitrary cross section is analyzed. Exact analytical solutions for flow profile of a channel flow with no-slip boundary conditions have been obtained in literature only for simple geometry of channel section. In this paper, a number of problems with more complicated geometries are solved either exactly or approximately. Three general solution methods are discussed, namely, complex function analysis, membrane vibration analogy and variational method. The usefulness of each method is justified with the help of examples.

1. Introduction

Microfluidics has become an active field of research following development of micro devices like micro sensors, micro mixers which find application in various fields of science and engineering [1]. The flow problem within a straight microchannel has been and still is a subject of research because in the micro level the flow shows significant deviation from that within a macrochannel. In the literature, the problem has been analytically solved only for a few simple cross sectional geometries. The geometry of a microchannel can, however, be complicated due to manufacturing restrictions. For example, the cross section may have Gaussian profile during laser ablation in the surface of polymer PMMA, the sidewalls of a rectangular channel may have wall slope etc. The aim of this paper is to describe various methods of analysis of fluid flow in a straight microchannel of arbitrary cross section. As will be shown, great many cases can be analyzed.

Three analytical methods are described in this paper. In the first method, functions of a complex variable are effectively used, whereas, the second method exploits the analogy of the problem with membrane vibration. In the third method, a variational formulation of the problem is given that can often aid in approximately calculating the velocity profile within a channel of arbitrary cross section.

In this paper, it is assumed that the pressure driven flow in microchannel is incompressible viscous flow governed by Navier–Stokes equation, where inertial forces can be neglected. No-slip boundary condition is assumed though for very narrow channels this boundary condition may not hold [2–4]. It is to be mentioned that although the analysis has been carried out for microchannels, same flow equations appear as well in macrochannels if the linear flow is assumed to be steady, fully developed and laminar [5].

2. Problem statement

Consider viscous incompressible flow within a straight microchannel of uniform cross section. The steady flow velocity $u(y, z)$ along axial direction is governed by the following equation

$$\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\mathrm{d}p}{\mathrm{d}x}, \tag{1}$$

where $\frac{\mathrm{d}p}{\mathrm{d}x}$ is the constant pressure gradient along the length axis x. The y- and z-axes are orthogonal to the length axis and x–y–z form a right handed co-ordinate system. No-slip condition at the boundary is given as

$$u(y,z)=0 \quad \text{at} \quad \varphi(y,z)=0, \tag{2}$$

* Tel.: +91 3222 282994.
E-mail address: goutam@mech.iitkgp.ernet.in

0017-9310/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2007.11.058

$$\text{or} \quad \frac{\partial u}{\partial s} = 0 \quad \text{along} \quad \varphi(y,z) = 0, \tag{3}$$

where s is the length measured along the boundary represented as $\varphi(y,z) = 0$. The boundary curve is assumed to be rectifiable.

So far, the exact velocity profile has been obtained for a limited number of cases, e.g., flow through circular or elliptical pipes, channel with rectangular cross section [6]. However, the form of the mathematical problem appears in solid mechanics, for example, the problem of torsion of a shaft with non-circular cross section [7], deflection of membrane under constant load. If, analogy with physical problems are considered then a number of problems can be solved either exactly or approximately. Three solution methods are discussed below.

3. Closed form solution using complex functions

In this section, an exact analytical solution technique using complex function analysis is given. The solution of Eq. (1) can be given as

$$u(y,z) = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)(y^2+z^2) + u_1(y,z), \tag{4}$$

where the function $u_1(y,z)$ satisfies the following equation

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} = 0. \tag{5}$$

It is well known that complex functional analysis can be used to solve Eq. (5). In fact, the solution is either the real part or the imaginary part of an analytic function $f(y+\mathrm{i}z)$ where $\mathrm{i} = \sqrt{-1}$. If the function f is chosen in such a way that $u(y,z)$ vanishes at the boundary, i.e., Eqs. (2) or (3) is satisfied then the corresponding $u(y,z)$ is the solution of the original problem. The method is explained with a few examples.

Example 1. To find the velocity profile of flow within a channel of circular cross section whose boundary is given by $\varphi(y,z) = y^2+z^2-a^2 = 0$, the complex function $f(y+\mathrm{i}z)$ is taken as a constant, say C. If $C = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)a^2$, then the flow profile becomes

$$u(y,z) = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)(y^2+z^2-a^2) \tag{6}$$

or in polar coordinate $u(y,z) = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)(r^2-a^2)$. This is the solution of the given problem as it satisfies the boundary condition.

Example 2. In this example the cross section is assumed to be in the form of an equilateral triangle. Let the sides of the triangle are represented as

$$y - \sqrt{3}z - \frac{2}{3}\alpha = 0, \tag{7}$$

$$y + \sqrt{3}z - \frac{2}{3}\alpha = 0 \tag{8}$$

and

$$y + \frac{1}{3}\alpha = 0. \tag{9}$$

In this problem assume $f(\xi) = a\xi^3 + b$ where $\xi = y + \mathrm{i}z$ with a, b as arbitrary constants. Taking the real part of f as $u_1(y,z)$, one gets,

$$u_1(y,z) = a(y^3 - 3yz^2) + b, \tag{10}$$

yielding

$$u(y,z) = \frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left[\frac{1}{2}(y^2+z^2) - \frac{1}{2\alpha}(y^3-3yz^2) + \beta\right], \tag{11}$$

where $a = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\frac{1}{\alpha}$ and $b = \frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\beta$. For the given problem setting $\beta = -\frac{2}{27}\alpha^2$, one finally gets

$$u = -\frac{1}{4\mu}\frac{1}{\alpha}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(y - \sqrt{3}z - \frac{2}{3}\alpha\right)\left(y + \sqrt{3}z - \frac{2}{3}\alpha\right)\left(y + \frac{1}{3}\alpha\right). \tag{12}$$

Eq. (12) is seen to produce the exact solution of the problem as it vanishes at the boundaries represented by Eqs. (7)–(9).

Example 3. Consider a channel with cross section shown in Fig. 1. The section is made of two circular arcs, one of a circle of radius $b$with origin at center and the other of circle of radius a with origin at $(a,0)$. In order to obtain flow profile the use of polar co-ordinates is most effective. The complex function is taken as $f(\xi) = A\xi + \frac{B}{\xi} + C$, where A, B and C are constants. Taking the real part of f one gets the solution of Eq. (5) as

$$u_1(r,\theta) = Ar\cos\theta + \frac{B}{r}\cos\theta + C, \tag{13}$$

where $r^2 = y^2 + z^2$ and $\tan\theta = z/y$. For the given problem choose $A = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right), B = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)2b^2a$ and $C = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)b^2$. The flow profile then becomes

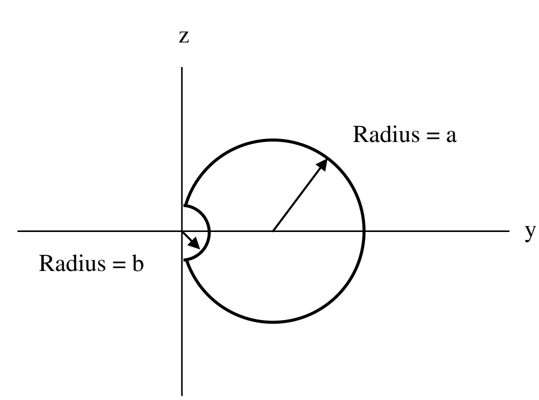

Fig. 1. Channel cross section made by two intersecting circles.