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Couette flow with frictional heating in a fluid with temperature and pressure dependent viscosity

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ABSTRACT

This study investigates the effects of variable viscosity and frictional heating on the laminar flow in a horizontal channel having a wall at rest and a moving wall subjected to a prescribed shear stress. The wall at rest is thermally insulated, while the moving wall is kept at a uniform temperature. This investigation concerns fluids whose viscosity depends exponentially on the pressure and temperature. An appropriate approximation is introduced to analyze the interplay between the dependence of viscosity on the pressure and temperature and the viscous dissipation. It is shown that the nonlinear term in the equation for the balance of energy representing the frictional heating may lead to the existence of dual solutions of the boundary value problem for fixed values of the material parameters that characterize the fluid. The results obtained are compared with those predicted by the generalization of the Oberbeck–Boussinesq approximation for a fluid with pressure and temperature dependent viscosity. It is found that the results for the approximation carried out in this paper and those that stem from the Oberbeck–Boussinesq approximation are markedly different.

1. Introduction

Under normal operating conditions, the viscosity of an incompressible fluid is assumed to be independent of the pressure. However, it is well known that the viscosity of a fluid can vary with pressure, and if the pressure range is significantly large the viscosity can change by several orders of magnitude. In his celebrated paper on the response of fluids, Stokes [35] recognizes that the viscosity of a fluid could depend upon the pressure. However, based on the experiments of Du Buat on the flow of water in canals and pipes under normal operating conditions, Stokes suggested that the viscosity could be considered a constant for such flows. Stokes is however very careful to delineate the class of flows wherein viscosity might be considered a constant and he also remarks that such an assumption would be invalid under other flow conditions. Unfortunately, the caveat that Stokes was careful to document has been largely ignored, both because of the mathematical ease that the assumption of constant viscosity confers, but also due to a lack of attention to detail of the subsequent researchers. In many organic liquids, while the density might change by a few percent due to a significant change in the pressure, the viscosity, on the contrary, could change by many orders of magnitude (as much as a factor of 10^8!) (see, for instance, [22] and references therein). Thus one could consider such liquids as incompressible fluids with pressure dependent viscosities. It is important to bear in mind, while making such an approximation that no body is truly incompressible and based on the process class that the body is subject to we can either choose to neglect or include the volume changes that occur.

As early as 1893 Barus et al. [3] proposed an empirical relationship between the viscosity and the pressure, namely

$$\mu(p) = \mu_{\text{ref}} \exp[\beta(p - p_{\text{ref}})],$$

where μ_{ref} is the viscosity at the reference pressure p_{ref} and β is a coefficient that determines how the viscosity varies with pressure, the coefficient depending upon the temperature. Later, Andrade [1] suggested the following expression for the viscosity

$$\mu(p,\rho,\theta) = A\rho^{1/2} \exp\left[(p + \rho r^2)\frac{s}{T}\right],$$

based on experiments. In the above expression ρ denotes the density, T the temperature, p the pressure, and r, s and A are constants. More recently, Laun [11] has modeled the viscosity of polymer melts through

$$\mu(p,T) = \mu_{\text{ref}} \exp[\beta(p - p_{\text{ref}}) - \gamma(T - T_{\text{ref}})], \quad (1)$$

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where μ_{ref} is the viscosity at the reference pressure p_{ref} and reference temperature T_{ref}, while β and γ are non-negative constants. There have been numerous other experiments by Bair and co-workers that shows that the dependence of the viscosity on the pressure is exponential (see the recent experiments of Bair and Kottke [2]). Mention must be made of the work of Martín-Alfonso and co-workers [17] wherein an intricate relationship among the temperature, viscosity and pressure is provided for bitumen. In this context, it ought to be pointed out that the pressure dependence of the properties of bitumen were recognized very early. For instance, Saal and Koens [32] not only allowed for viscosity to depend on pressure (the normal stress), they even allowed it to depend on the shear stresses. Thus, they had a truly implicit constitutive model relating the stress and kinematical quantities (see also [18,33]).

The Oberbeck–Boussinesq approximation is one of the most useful approximations in fluid mechanics and is used extensively to study problems in astrophysical and geophysical fluid dynamics. However, the approximation is not an approximation that is a perturbation in which like terms of the same order in a small parameter are retained. Until recently, many attempts were been made to justify the approximation but none of them are compelling and lack rigor (the reader is referred to [24] for a detailed discussion of these attempts). The main problem with all these attempts is the necessity to retain a term in the equations that is a product of the coefficient of thermal expansion and gravity is to be of order unity while the coefficient of thermal expansion has to tend to zero. The problem stems from the small parameter that is used in the perturbation. Recently, Rajagopal et al. [24] have introduced a different parameter, one suggested by Chandrasekhar [5], to carry out the perturbation and have systematically derived the Oberbeck–Boussinesq approximation. In the paper, Rajagopal et al. [24] introduce a different method to deal with the constraint of a body being mechanically incompressible but having the ability to change volume due to temperature changes. The key to a proper basis for the Oberbeck–Boussinesq approximation in the choice of the small parameter with which to carry out the perturbation and once a proper choice is made, the rest of the analysis follows easily. The classical Oberbeck–Boussinesq approximation concerns a fluid with constant viscosity and thus when the viscosity of the fluid is not constant and depends on variables such as the temperature and pressure, one needs to develop an approximation that is appropriate for such fluids.

In recent years there has been a considerable interest in fluids with pressure dependent viscosity. Rajagopal and co-workers ([8,15,16,21–23,25–27,36,38]) have studied in detail several important and basic problems related to such fluids. In particular, in [26] we have provided a rigorous mathematical justification for the Oberbeck–Boussineq approximation that appeals to a perturbation procedure. The small parameter ϵ with respect to which we have performed our perturbance is the product of the coefficient of thermal expansion evaluated at the reference temperature and the difference between the constant temperatures at which the walls bounding the horizontal fluid layer are kept. We have derived the Oberbeck–Boussineq approximation for fluids whose material moduli are analytical functions of pressure and temperature by assuming the Froude number to be of order $O(\sqrt{\epsilon})$. By virtue of this assumption the viscous dissipation term in the equation of balance of energy may be neglected. But there are several problems concerning flows of fluids in ducts in which the effects of viscous dissipation is not negligible, which is a necessary consequence of the Oberbeck–Boussineq approximation. The inclusion of viscous dissipation is particularly important when dealing with highly viscous fluids such as engine oils and polymer melts.

In the literature there is a significant amount of theoretical and numerical results concerning the subject of laminar convection with frictional heating (see for instance [3,4,6,7,10,12–14,19,20,34] and references therein) due to their relevance to several engineering applications. In these works the authors automatically invoke the Oberbeck–Boussinesq approximation in developing the equation that stems from the balance of momentum while at the same time retaining the viscous dissipation term in the equation for the balance of energy. Dealing with these non linear governing equations leads to non-existence or non-uniqueness of the solution depending on the choice of the boundary conditions. Examples of multiple laminar solutions are discussed in [3,4,6,10,12]. In this paper we shall not invoke the Oberbeck–Boussinesq approximation in the equation that stems from the balance of linear momentum as seems to be customary in the literature until now, but we shall introduce a new set of approximate equations adapted to investigate the flows in highly viscous fluids wherein the frictional heating is not negligible. The approximation we shall introduce in Section 3 differs from the equations adopted in the literature only in the equation for the balance of linear momentum (see Eq. (39) and note that the fluid density which appears is the constant reference density).

The plan of the paper is the following. In Section 2 we introduce the constitutive fluid model and derive the governing equations. In Section 3 we derive the approximate equations by using a perturbation procedure. The small parameter with respect to which we shall carry out our perturbance will be the dimensionless parameter ϵ mentioned above, under the assumption that the Froude number is now of order of unity or greater so that the viscous dissipation term in the equation for the balance of energy may be retained. In Section 4 we shall analyze the steady fully-developed mixed convection in a horizontal plane parallel channel filled with a fluid whose viscosity depends on pressure and temperature according to (1) and with a wall at rest and thermally insulated. The other wall is kept at constant temperature and subjected to a uniform shear stress generating a uniform wall velocity. The results obtained reveal that, for given values of the material moduli that appear in the constitutive representation for the fluid, the solution of the boundary value problem does not always exists and when it does exist, it is in general not unique. Finally, these results are compared with the solution to the corresponding boundary-value problem arising from the generalized Oberbeck–Boussinesq equations we have derived in [26].

2. Governing equations

We record below the local forms of the balance of mass, linear momentum and energy, as well as the second law of thermodynamics in the form of the Clausius–Duhem inequality (see, for instance, [37]):

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \tag{2}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{3}$$

$$\rho \dot{e} + \operatorname{div} \mathbf{q} = \mathbf{T} \cdot \mathbf{D} + \rho r, \tag{4}$$

$$\rho \dot{\eta} \geqslant \rho \frac{r}{\theta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right), \tag{5}$$

where the dot denotes total time derivative, ρ is the density, $\mathbf{v}$ the velocity field, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{b}$ the specific external body force field, e the specific internal energy, r the specific radiant heating, θ the temperature, $\mathbf{q}$ the heat flux vector, $\mathbf{D}$ the symmetric part of the velocity gradient and η the specific entropy. A few words concerning the Clausius–Duhem inequality are warranted. The inequality has been used extensively to obtain restrictions on the allowed forms of the constitutive equations. One does this by subjecting the body to arbitrary processes and requiring that the second law be met in all these processes. This leads to necessary and sufficient conditions which will guarantee that the constitutive relations will meet the second law. However, one has to keep in mind that the constitutive relations that one chooses are predicated