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Transient heat conduction in a medium with two circular cavities: Semi-analytical solution

Elizaveta Gordeliy *, Steven L. Crouch, Sofia G. Mogilevskaya

Department of Civil Engineering, University of Minnesota, Minneapolis, MN 55455, USA

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Abstract

This paper considers a transient heat conduction problem for an infinite medium with two non-overlapping circular cavities. Suddenly applied, steady Dirichlet type boundary conditions are assumed. The approach is based on superposition and the use of the general solution to the problem of a single cavity. Application of the Laplace transform results in a semi-analytical solution for the temperature in the form of a truncated Fourier series. The large-time asymptotic formulae for the solution are obtained by using the analytical solution in the Laplace domain. The method can be extended to problems with multiple cavities and inhomogeneities.

Keywords: Transient heat conduction; Laplace transform; Modified Helmholtz equation; Fourier series; Addition theorem; Asymptotic series

1. Introduction

This paper presents a semi-analytical solution for a transient heat conduction problem for an infinite medium containing two circular cavities. This problem occurs in several engineering applications, for example, heat exchange between the earth and buried pipes [1], cooling of tunnels [2], and heat exchange between blood tissue and embedded blood vessels [3]. The problem is also of interest for modeling time-dependent effects due to diffusion processes, such as unsteady fluid flow [4,5].

As in many other applications, the use of analytical solutions in transient heat conduction problems is very beneficial. Such solutions can be used to study possible singularities, to obtain accurate solution gradients (e.g. heat fluxes), as well as the asymptotic approximations for the solutions for small and large values of time. In addition, knowledge of analytical solutions can provide benchmark results to test newly developed numerical methods.

The method of solution presented here for a problem of two circular cavities is based on the use of the analytical solution to a corresponding problem of a single cavity and superposition. The single cavity problem has been extensively studied and various particular solutions are available in the literature (e.g. [6]). Analytical and semi-analytical solutions for the case of multiple cavities are available only for the steady-state case (e.g. [3,6]).

Transient problems with cavities can be solved by general purpose numerical methods such as finite element, finite difference, and boundary element methods combined with time-marching schemes. For large-time computations these approaches can be computationally intensive due to time-marching and large numbers of degrees of freedom. To efficiently treat the time convolution involved in the problem several fast numerical techniques have been recently developed (see e.g. [7–9] and references therein).

A number of numerical methods based on the use of the Laplace transform (or Fourier transform) have also been designed to solve transient problems. In such methods the original transient problem is transformed to a corresponding non-transient problem in the Laplace domain (or frequency domain), which is easier to solve. After the

* Corresponding author. Tel.: +1 612 626 8768; fax: +1 612 626 7750.
E-mail address: gord0232@umn.edu (E. Gordeliy).

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Nomenclature

$\binom{m}{n}$ binomial coefficient, $\binom{m}{n} = \frac{m!}{n!(m-n)!}$
a_k intermediate variable, $a_k = r_k/R_k$
$\mathbf{a}^k(x,s)$ (N_k+1)-dimensional vector, Eq. (23)
$\mathbf{a}^k_{pm}(x)$ vector coefficients of the asymptotic expansion of $\mathbf{a}^k(x,s)$, Eq. (31)
$A_{pm}(x)$ coefficients of the asymptotic expansion of $\widehat{T}(x,s)$, Eq. (29)
$A^{kl}_n(s)$, $B^{kl}_n(s)$ Fourier coefficients of the boundary value $\widehat{T}_l(x,s)|_{L_k}$ $(k \neq l)$, Eq. (13)
$\mathbf{A}^{kl}(s)$ (N_k+1)-dimensional vector of Fourier coefficients $A^{kl}_n(s)$, Eq. (14)
$B^{p,i}_{m-i}$ intermediate coefficients, Appendix D
$\mathbf{B}^{kl}(s)$ N_k-dimensional vector of Fourier coefficients $B^{kl}_n(s)$, Eq. (14)
$\mathbf{b}^k(x,s)$ N_k-dimensional vector, Eq. (23)
$\mathbf{b}^k_{pm}(x)$ vector coefficients of the asymptotic expansion of $\mathbf{b}^k(x,s)$, Eq. (32)
c^k_n, d^k_n Fourier coefficients of the function $\Phi_k(\varphi_k)$, Eq. (5)
$\mathbf{c}^k$ (N_k+1)-dimensional vector of Fourier coefficients c^k_n, Eq. (14)
$\mathbf{d}^k$ N_k-dimensional vector of Fourier coefficients d^k_n, Eq. (14)
$\mathbf{F}^{kl}(s)$ $(N_k+1)\times(N_l+1)$-dimensional matrix $(k \neq l)$, Eq. (15)
$\mathbf{F}^{kl}_{pm}$ matrix coefficients of the asymptotic expansion of $\mathbf{F}^{kl}(s)$, Eq. (64)
$f(u)$ integrand matrix-function, Section 6.2
$\mathbf{G}^{kl}(s)$ $N_k \times N_l$-dimensional matrix $(k \neq l)$, Eq. (15)
$\mathbf{G}^{kl}_{pm}$ matrix coefficients of the asymptotic expansion of $\mathbf{G}^{kl}(s)$, Eq. (65)
H_k steady-state flux, Eq. (28)
$\mathbf{I}_N$ $N \times N$-dimensional identity matrix
$I_n(\cdot)$, $K_n(\cdot)$ modified Bessel functions [21]
k, l number of the cavity, $k = 1,2$ and $l = 1,2$
L_k boundary of the kth cavity
M number of steps in the alternating algorithm, Eqs. (51) and (52)
M_0, M_1 numbers of terms in asymptotic series (29) and (44)
$\mathbf{m}^k_{0,-1}$ $1\times(N_k+1)$-dimensional matrix, Appendix C
N_k number of terms in the truncated Fourier series, Eq. (10)
$\mathbf{n}^k$ annihilating vector, Section 5.1
q transform variable, $q = \sqrt{s}$
R_k dimensionless ratio of the radius of the kth cavity to the distance ρ
r_k dimensionless radial polar coordinate = ratio of the distance between point x and the center of the kth cavity to the distance ρ
$\mathbf{R}^k_{npm}$ right-hand side matrices in Eq. (82)
s Laplace transform parameter
t dimensionless time, Eq. (1)
t_1 minimum specified time instant, Section 6
t_n dimensionless time, at which the solution is computed, Section 6
$T(x,t)$ dimensionless temperature, Eq. (1)
$T_s(x)$ steady-state temperature, Eqs. (26) and (27)
$\widehat{T}(x,s)$ Laplace transform of $T(x,t)$
$\widehat{T}_k(x,s)$ solution to the Laplace-transformed problem containing only the kth cavity, Eq. (9)
u integration variable, Eq. (26)
$\mathbf{U}^k(s)$ $(N_k+1)\times(N_k+1)$-dimensional matrix, Eq. (19)
$\mathbf{U}^k_{pm}$ matrix coefficients of the asymptotic expansion of $\mathbf{U}^k(s)$, Eq. (37)
$\widetilde{\mathbf{U}}^k_{pm}$ matrix coefficients of the asymptotic expansion of $[\mathbf{U}^k(s)]^{-1}$, Eq. (34)
$\mathbf{u}^k(s)$ (N_k+1)-dimensional vector, Eq. (19)
$\mathbf{u}^k_{pm}$ vector coefficients of the asymptotic expansion of $\mathbf{u}^k(s)$, Eq. (75)
$\mathbf{V}^k(s)$ $N_k \times N_k$-dimensional matrix, Eq. (20)
$\mathbf{V}^k_{pm}$ matrix coefficients of the asymptotic expansion of $\mathbf{V}^k(s)$, Eq. (38)
$\widetilde{\mathbf{V}}^k_{pm}$ matrix coefficients of the asymptotic expansion of $[\mathbf{V}^k(s)]^{-1}$, Eq. (35)
$\mathbf{v}^k(s)$ N_k-dimensional vector, Eq. (20)
$\mathbf{v}^k_{pm}$ vector coefficients of the asymptotic expansion of $\mathbf{v}^k(s)$, Eq. (76)
x point in the two-dimensional domain
x_m point in the two-dimensional domain, at which the solution is computed, Section 6
$Y^k_n(s)$, $Z^k_n(s)$ unknown Fourier coefficients, Eq. (10)
$\mathbf{Y}^k(s)$ (N_k+1)-dimensional vector of unknowns, Eq. (14)
$\mathbf{Z}^k(s)$ N_k-dimensional vector of unknowns, Eq. (14)

Greek symbols
α_k, β_k, β intermediate constants, Eq. (55)
γ Euler's constant, $\gamma = 0.5772\ldots$
δ intermediate integration limit, Eq. (50)
δ_{ij} Kronecker delta symbol
ε predefined accuracy level, Eq. (47)
$\Theta(x,\tau)$ temperature at point x at time τ
Θ_0 uniform initial temperature, $\Theta_0 = \Theta(x,0)$
κ constant thermal diffusivity
$\Lambda_{pm}(x)$ coefficients of the large-time asymptotic series, Eq. (44)
μ_k scalar factor, Appendix C
ρ distance between the centers of the cavities
τ time