International Journal of Heat and Mass Transfer 54 (2011) 180–185

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Critical insulation thickness of a rectangular slab embedded with a periodic array of isothermal strips

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ARTICLE INFO

Article history:
Received 16 April 2010
Received in revised form 16 September 2010
Accepted 16 September 2010
Available online 31 October 2010

Keywords:
Solid slab with periodic array of isothermal strips
Heat conduction/convection
Critical insulation thickness
Shape factor
Total thermal resistance
Overall heat transfer coefficient
Laplace equation

ABSTRACT

We address the problem of two-dimensional heat conduction in a solid slab embedded with a periodic array of isothermal strips. The surfaces of the slab are subjected to a convective heat transfer boundary condition with a uniform heat transfer coefficient. Similar to the concept of critical insulation radius, associated with cylindrical and spherical configurations, we show that there exists a critical insulation thickness, associated with the slab, such that the total thermal resistance attains a minimum, i.e. a maximum heat transfer rate can be achieved. This result, which is not observed in one-dimensional heat conduction in a plane wall, is a consequence of the non-trivial coupling between conduction and convection that results in a 2D temperature distribution in the slab, and a non-uniform temperature on the surface of the slab. The findings of this work offer opportunities for improving the design of a broad range of engineering processes and products.

1. Introduction

When an isothermal planar surface is covered with insulation, the total thermal resistance is always increased, and the effect is to reduce the energy dissipation [1,2]. This result follows logically as one expects that the increase in the conduction path leads to an increase in the total resistance. However, in the case of cylindrical and spherical systems, adding this layer of insulation also increases the surface area available for heat transfer by convection. As there are two competing mechanisms that control the total rate of heat transfer, one expects that there is a critical radius such that the heat transfer rate is maximized. Indeed, it is a simple exercise to show that for cylindrical and spherical systems such a critical radius, where the total thermal resistance attains a minimum, exists and it is known as the critical insulation thickness [1]. This result is also applicable for surfaces with variable convection coefficient, radiation loss and cylinders with transparent insulation [4–6]. The ideas can be also extended to non-circular domains: (i) square and rectangular domains etc. [2] where, using the shape factor, it is shown that a more general criterion for maximum heat dissipation is the critical perimeter of insulation, and (ii) domains with eccentric circular insulation where the maximum heat loss and the corresponding optimum insulation configuration must be found from the solution of the two-dimensional heat conduction equation with convective boundary condition imposed on the outside surface of the insulation [3]. In this work, similar to the latter approach [3], we show that the concept of critical insulation thickness can be also established for a planar, non-isothermal surface. The temperature variation is due to a finite, isothermal strip embedded below the surface that creates a non-trivial interaction between conduction through the slab and convection heat transfer from the planar surface.

Heat transfer in the slab-like configurations addressed here, is of interest in systems with distributed energy sources. In particular, high performance computing (e.g. laptops and desktop computers) requires multiple CPU units, each of which is a significant source of thermal energy. Thermal management in these systems is a topic of active research [7–9]. Moreover, both the problem formulated in this work and the results are relevant to a class of manufacturing processes related to thermal processing or operation of layered structures: (i) self-curing/bonding of laminate polymer matrix composites (PMC) where the heat is produced internally by conductive strips [10–13], and (ii) internal (self) rapid thermal processing of semiconductor structures through embedded strips of nanoheaters [14,15]. Besides heat transfer, many problems in modern science involve the solution of the Laplace equation,

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doi:10.1016/j.ijheatmasstransfer.2010.09.056

Nomenclature

Bi	Biot number $Bi = hW/k$ (dimensionless)	T	temperature (K)
G	Green's function of the temperature field (dimensionless)	U	overall heat transfer coefficient (W/m^2/K)
H	one half the thickness of the slab (m)	W	width of a strip (m)
h	convection heat transfer coefficient (W/m^2/K)	x, y	coordinates of the physical plane (m)
k	thermal conductivity (W/m/K)	z	span of the strip (m)
L	distance between two consecutive strips (m)		
R_{tot}	total thermal resistance (K/W)	*Diacritic*	
S	shape factor (m)	$\wedge$	the variable is normalized with the width of the strip W

hence, the results of this study offer opportunities for improving the design of a broad range of engineering processes and products.

In the next Section 2 we formulate an integral equation associated with the two-dimensional, heat conduction problem associated with a slab embedded with a periodic array of finite isothermal strips, and we obtain a Fredholm integral equation for the temperature gradient along a strip. The integral equation is solved both analytically (asymptotically) and numerically. The latter is discussed in Section 3. The results are also verified through a finite-element numerical calculation. We summarize our findings in Section 4.

2. Problem formulation and asymptotic results

Consider two-dimensional heat conduction due to a periodic array of isothermal (T_1) strips of infinite span, embedded at the center of a solid slab. The slab is subjected to convection heat transfer along its upper and lower surfaces as shown in Fig. 1. The strips of length W are placed a distance L apart. We non-dimensionalize lengths with the length of the strips (W), and the temperature field by subtracting T_∞ and normalizing by the temperature difference ($T_1 - T_\infty$). All non-dimensional variables are denoted by $\wedge$. In addition, because of symmetry, we consider only the upper half of the region, hence the formulation is also relevant for a periodic array of isothermal strips embedded in a non-conducting substrate [16].

At steady-state, the temperature distribution is governed by the Laplace equation $\nabla^2 \widehat{T} = 0$. In view of the symmetry of the problem in the $\hat{y}$ direction and periodicity in the $\hat{x}$ direction, the boundary conditions are:

$$\text{On } \hat{y} = 0 \begin{cases} \widehat{T}[\hat{x}] = 1 & \text{along } 0 \leqslant \hat{x} \leqslant 1, \\ \dfrac{\partial \widehat{T}}{\partial \hat{y}}[\hat{x}] = 0 & \text{along } 1 < \hat{x} < \widehat{L}, \end{cases}$$
$$\text{On } \hat{y} = \widehat{H} \quad \frac{\partial \widehat{T}}{\partial \hat{y}}[\hat{x}] + Bi\, \widehat{T}[\hat{x}] = 0,$$
$$\widehat{T}[\hat{x} = 0, \hat{y}] = \widehat{T}[\hat{x} = \widehat{L}, \hat{y}], \tag{1}$$

where $Bi = hW/k$ defines the Biot number. The mathematical model along with the boundary conditions are shown in Fig. 2. The solution depends on the two geometric parameters, $\widehat{H}$ and $\widehat{L}$, and the Biot number (Bi).

In the next section, using Green's theorem [17–19], we obtain a Fredholm integral equation of the first kind for the temperature gradient along a single strip. This equation is solved numerically, and asymptotic results for some limiting cases are also developed.

2.1. Green's theorem formulation and asymptotic results

The objective of this section is to determine, as a function of $\widehat{L}$, $\widehat{H}$ and Bi, the heat transfer rate (transport rate) from a single strip. Equivalently, we define the dimensionless shape factor (S) [20], the total thermal resistance (R_{tot}) and the overall heat transfer coefficient (U) [1] associated with a single strip as:

$$S = -\mathrm{z} \int_0^1 \frac{\partial \widehat{T}[\hat{x}, \hat{y} = 0]}{\partial \hat{y}} \mathrm{d}\hat{x},$$
$$R_{tot} = \frac{1}{Sk},$$
$$U = \frac{Sk}{L\,\mathrm{z}}, \tag{2}$$

respectively [1], where z is the dimensional span of the strip. Note that we only consider the upper-half of the domain because of symmetry.

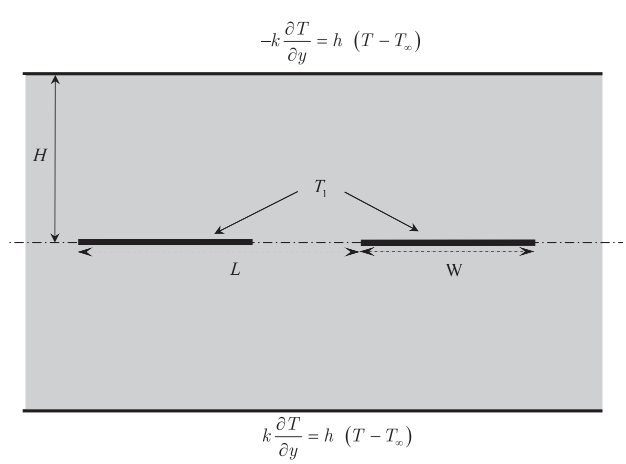

Fig. 1. Schematic representation of the physical problem. The surfaces of the slab are subject to uniform convection, characterized by a heat transfer coefficient h, and the distance between them is $2H$. At the mid-distance there is a two-dimensional, periodic array of isothermal strips. The width of the strips is W and the distance between two successive strips is L. The strips are kept at temperature T_1.

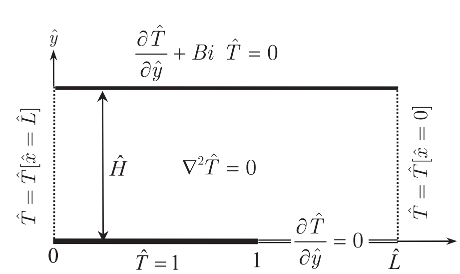

Fig. 2. Schematic representation of the model problem along with boundary conditions. The Neumann boundary is dictated by the symmetry of the configuration.