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The onset of convection in a porous layer induced by viscous dissipation: A linear stability analysis

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ABSTRACT

The effect of viscous dissipation on parallel Darcy flow in a horizontal porous layer with an adiabatic lower boundary and an isothermal upper boundary is discussed. The presence of viscous dissipation serves to cause a nonlinear temperature profile within the layer. The linear stability of this nonisothermal base flow is then investigated with respect to the onset of convective rolls. The solution of the linear equations for the perturbation waves is determined analytically by a power series method, and the results are confirmed using a direct numerical approach using a fourth order Runge Kutta method. The neutral stability curve and the critical value of the governing parameter $R = Ge\,Pe^2$ are obtained, where Ge is the Gebhart number and Pe is the Péclet number. The effect of an imperfect isothermal boundary condition at the upper boundary is investigated by considering finite values of the Biot number.

1. Introduction

The onset of convection in a horizontal fluid-saturated porous layer heated from below has been widely studied in the last few decades. The interest in this subject is related both to geophysical research and to engineering design. Possible applications include the analysis of water currents in a porous rock, the underground spread of pollutants, the enhancement of the performance in building insulation, solar energy collectors and solar ponds. Wide and detailed discussions of the literature on this subject can be found in the book by Nield and Bejan [1] and the reviews of Rees [2] and Tyvand [3].

The early papers of Horton and Rogers [4] and Lapwood [5] presented the first linearised stability analyses of what has become known widely as either the Horton–Rogers–Lapwood (HRL) problem or the Darcy–Bénard (DB) problem. The former links us to the pioneers of stability theory in porous media, while the latter emphasizes the strong link with the Rayleigh-Bénard problem. The classical DB problem consists of a basic motionless state with a uniform temperature drop across the layer with warmer fluid lying below cooler fluid.

Very many authors have developed variants of this basic stability problem either by employing porous models that are more complicated than Darcy's law, or by altering the external conditions, such as imperfectly conducting boundaries or the presence of internal heating, rotation or vertical throughflow. Of most interest here is the study of Prats [6] who investigated the effect of a uniform parallel basic flow in the layer which might be caused by applying a uniform horizontal pressure along the layer. By using a moving frame of reference Prats proved that this uniform basic flow does not alter the condition for the onset of instability. In Prats' treatment, the critical value of Rayleigh number, Ra_{cr}, is the same as in the DB problem, viz. $4\pi^2$. Moreover, the full nonlinear equations, when written in the moving frame, reduce to those which apply when there is no basic flow. Therefore, the full nonlinear behaviour of the DB problem is recovered in an infinitely long layer. One other consequence is that there is no preferred direction for the roll orientation at onset, a property which it does not share with Bénard–Poiseuille convection.

With regard to what we shall call the Darcy–Bénard–Prats (DBP) problem, there exist some recent papers which have extended the work of Prats [6]. Rees [7] considered the effect of quadratic form drag in the momentum equation. He showed that the critical Darcy–Rayleigh number, Ra_{cr}, depends on both the form drag coefficient and on the base flow velocity. Moreover, the critical Darcy–Rayleigh number is also dependent on the roll orientation, with longitudinal rolls forming the preferred pattern. The additional effects of lateral confinement were considered by

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Nomenclature

a	nondimensional wave number, Eq. (28)
A_n	nth series coefficient, Eq. (36)
Bi	Biot number, hL/k
c_p	specific heat at constant pressure
$c_{\rm wave}$	nondimensional phase velocity, Eq. (35)
g	modulus of gravitational acceleration
$\mathbf{g}$	gravitational acceleration
G	nondimensional parameter, $Ge(\cos\chi)^2$
Ge	Gebhart number, Eq. (13)
h	external heat transfer coefficient
K	permeability
k	thermal conductivity
L	channel height
$\mathscr{L}_n$	differential operator, Eqs. (A11), (A12)
n	integer number
P	nondimensional parameter, $Pe/\cos\chi$
Pe	Péclet number, Eq. (16)
R	nondimensional parameter, $GePe^2$
$\Re$	real part
$\mathbf{s}$	unit vector parallel to the base flow direction
t	nondimensional time, Eq. (7)
T	nondimensional temperature, Eq. (7)
T_w	upper boundary temperature or external temperature
u, v, w	nondimensional velocity components, Eq. (7)
U, V, W	nondimensional velocity disturbances, Eq. (17)
$\bar{\mathbf{u}}_B$	base flow velocity
x, y, z	nondimensional coordinates, Eq. (7)

Greek symbols

α	thermal diffusivity
β	volumetric coefficient of thermal expansion
γ	reduced exponential coefficient, Eq. (29)
ϵ	nondimensional parameter, Eq. (A3)
θ	nondimensional temperature disturbance, Eq. (17)
$\Theta(y)$	nondimensional function, Eq. (28)
λ	exponential coefficient, Eq. (28)
λ_1, λ_2	real and imaginary parts of λ
ν	kinematic viscosity
ρ	mass density
σ	heat capacity ratio
χ	angle between base flow direction and x-axis
ψ	nondimensional streamfunction, Eq. (24)
$\Psi(y)$	nondimensional function, Eq. (28)

Superscript, subscripts

–	dimensional quantity
B	base flow
cr	critical value

Delache, Ouarzazi and Néel [8]; these authors found discontinuous transitions between preferred roll states. Postelnicu [9] extended the work of Rees [7] by combining it with the work of Banu and Rees [10], who employed the two-temperature model for heat conduction. This model involves an inter-phase heat transfer coefficient to account for the absence of local thermal equilibrium between the solid and fluid phases. A comprehensive set of results is presented by Postelnicu [9] showing the detailed effect on the critical Darcy–Rayleigh number and wavenumber of the inertia parameter, the flow rate and the three parameters that are associated with local thermal nonequilibrium.

The aim of the present paper is to consider the following variant on the DBP problem. In the above-cited works thermoconvective instability was driven by an unstable temperature gradient that is imposed externally. In the present paper we shall assume that there is no imposed temperature gradient across the layer, but rather that heat is generated internally by the action of viscous dissipation. In particular the upper surface will be taken to be isothermal (infinite-Biot number), while the lower surface is thermally insulated. The former boundary condition is relaxed later in the paper by using a finite-Biot-number condition to represent external heat transfer to the ambient temperature.

A linear stability analysis of oblique rolls which are orientated arbitarily with respect to the uniform base flow direction is performed. The disturbance equations are solved both analytically by a series method and numerically by a fourth order Runge Kutta method. We present information on how the critical Darcy–Rayleigh number and wavenumber vary with the Gebhart and Péclet numbers. Asymptotic expressions for the critical quantities vs the Péclet number are obtained.

2. Mathematical model

We shall consider laminar buoyant flow in a horizontal parallel channel with height L (see Fig. 1). Both the Darcy model and the Boussinesq approximation are invoked. The components of seepage velocity along the $\bar{x}$-, $\bar{y}$- and $\bar{z}$-directions are denoted by $\bar{u}$, $\bar{v}$ and $\bar{w}$, respectively. The lower boundary wall $\bar{y} = 0$ is assumed to be adiabatic, while the upper boundary wall $\bar{y} = L$ is supposed to be isothermal with temperature T_w. Both boundary walls are impermeable. Later in the paper we relax the assumption of having a perfectly conducting upper boundary.

The governing mass, momentum and energy balance equations can be expressed as

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \tag{1}$$

$$\frac{\partial \bar{v}}{\partial \bar{x}} - \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{g\beta K}{\nu}\frac{\partial \bar{T}}{\partial \bar{x}}, \tag{2}$$

$$\frac{\partial \bar{v}}{\partial \bar{z}} - \frac{\partial \bar{w}}{\partial \bar{y}} = \frac{g\beta K}{\nu}\frac{\partial \bar{T}}{\partial \bar{z}}, \tag{3}$$

$$\frac{\partial \bar{u}}{\partial \bar{z}} - \frac{\partial \bar{w}}{\partial \bar{x}} = 0, \tag{4}$$

$$\sigma\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{T}}{\partial \bar{z}} = \alpha\left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{z}^2}\right) + \frac{\nu}{Kc_p}\left(\bar{u}^2 + \bar{v}^2 + \bar{w}^2\right), \tag{5}$$

where σ is the ratio between the average volumetric heat capacity $(\rho c_p)_m$ of the porous medium and the volumetric heat capacity $(\rho c_p)_f$ of the fluid. Eq. (2) has been obtained by combining the $\bar{x}$-component and the $\bar{y}$-component of Darcy's law in order to remove the explicit dependence on the pressure field; Eqs. (3) and (4) were obtained in a similar manner.

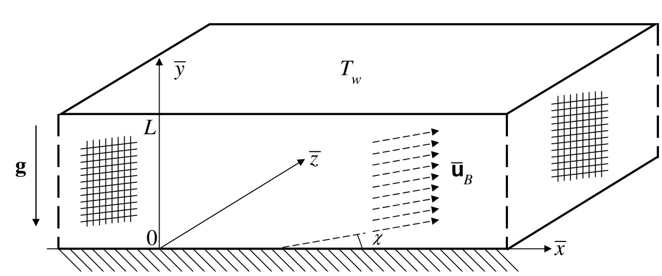

Fig. 1. Sketch of the horizontal porous channel.