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An approximate solution method for boundary layer flow of a power law fluid over a flat plate

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ABSTRACT

The work in this paper deals with the development of momentum and thermal boundary layers when a power law fluid flows over a flat plate. At the plate we impose either constant temperature, constant flux or a Newton cooling condition. The problem is analysed using similarity solutions, integral momentum and energy equations and an approximation technique which is a form of the Heat Balance Integral Method. The fluid properties are assumed to be independent of temperature, hence the momentum equation uncouples from the thermal problem. We first derive the similarity equations for the velocity and present exact solutions for the case where the power law index $n = 2$. The similarity solutions are used to validate the new approximation method. This new technique is then applied to the thermal boundary layer, where a similarity solution can only be obtained for the case $n = 1$.

1. Introduction

Describing the flow of a Newtonian fluid in the boundary layer above a flat plate is one of the classical problems of fluid mechanics. Since the majority of practical fluids are non-Newtonian the extension of this theory to such fluids is obviously also a key problem. Hence, in this paper the flow of a power law fluid past a flat plate, as well as the associated heat transfer, is examined.

The boundary layer flow of a power law fluid has received much analytical attention, see [6–8,10–12,16] for example. When dealing with the momentum boundary layer alone the problem may be analysed using similarity methods. For the Newtonian case the governing equations reduce to the Blasius equation: an ordinary differential equation that is easily solved numerically [26]. For a power law fluid the reduction of the system via similarity variables leads to a modified version of the Blasius equation [11,12,25,27].

When the thermal boundary layer is included, due to the differences in the power of the stress gradient and second derivative of temperature, a similarity solution is not possible (except in the Newtonian case). In this case there are two standard ways forward. The governing equations can be solved numerically, see [2,14,28] for example, or via integral methods (which will be discussed in detail later), see [1,6–8]. The accuracy of the latter approach is known to deteriorate as the fluid becomes less Newtonian, [8,12]. As discussed by Chhabra [8] the numerical results are more accurate than the integral methods but the integral methods are useful since they often lead to closed form solutions. For this reason in the following work we will examine the integral method approach, with a view to improving its accuracy.

In Section 2 we derive the governing equations and corresponding integral forms describing the momentum and thermal boundary layers. In Section 3 we discuss the similarity solutions for the original and integral forms of the momentum equation. It is shown that both problems have an exact solution for the case where the power law index $n = 2$. The numerical solution of the appropriate ordinary differential equations tends to these solutions as $n \to 2$. In Section 4 we describe the standard approximation to the momentum equations attributed to von Kármán and Pohlhausen, see [26], as well as a more accurate method developed by Chhabra [7,8]. We then demonstrate a variation of the method designed for the analogous Heat Balance Integral Method that minimises the error introduced by solving the governing equations only in an integral sense, see [18,21–23]. After demonstrating the improved accuracy of the new method we then apply it to the thermal boundary layer in Section 5 for a constant temperature, a constant flux and Newton cooling condition at the plate.

2. General theory

The boundary layer equations for two dimensional steady incompressible flow are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty\frac{dU_\infty}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

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Nomenclature

C	drag coefficient
$H = L/Re^{1/(n+1)}$	height scale
$\mathscr{H}$	convective heat transfer coefficient
L	plate length
m	flow consistency index
n	power law index
Re	Reynolds number $Re = \rho U_\infty^{2-n} L^n / m$
Pr	Prandtl number $Pr = H^2 U_\infty/(\kappa L)$
Q	non-dimensional heat flux at $y = 0$
T	fluid temperature
T_∞	far field temperature
$\mathbf{u} = (u, v)$	velocity vector
U_∞	far field velocity
$\delta(x),\ \delta_T(x)$	momentum and thermal boundary layer thickness
$\epsilon = \delta_T/\delta$	ratio of boundary layer thicknesses
κ	thermal diffusivity
ρ	fluid density
τ	shear stress
ξ	similarity variable

Subscripts

0	value in fluid at substrate $y = 0^+$
s	value at substrate $y = 0^-$

where (u, v) is the fluid velocity, U_∞ is the velocity in the far field and τ is the shear stress. The velocity profile is subject to the boundary conditions

$$u = v = 0 \tag{3}$$

at $y = 0$ and

$$u = U_\infty \tag{4}$$

as $y \to \infty$. At $x = 0$ the flow is the far field flow, $u(0, y) = U_\infty$. For a power law fluid we can set

$$\tau = m\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}, \tag{5}$$

where m is the consistency index and $n > 0$.

If the physical properties of the fluid only depend weakly on the temperature (so that we can assume they are constant) then the momentum boundary layer can be analysed independently of the thermal problem. The thermal problem depends on the flow and for an incompressible fluid is governed by

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \kappa\frac{\partial^2 T}{\partial y^2}, \tag{6}$$

where κ is the thermal diffusivity and $T \to T_\infty$ as $y \to \infty$, $T(0, y) = T_\infty$. We will discuss the boundary condition at $y = 0$ later.

For a Newtonian fluid equations (1)–(6), with the temperature specified at the plate $T(x, 0) = T_s$, can be examined using a similarity variable, see [9, p311]. However, when $n \neq 1$ the similarity reduction is not possible so we must resort to numerical or approximate solution methods.

A standard approximation is known as the Integral Momentum Equation (IME), [7]. The IME may be obtained from the boundary layer equations (1, 2)or via a simple mass and momentum balance argument, see [7, pp. 345–351], [26, p. 191]. Integrating Eq. (1) over $y \in [0, h]$, where h is everywhere greater than the boundary layer thickness leads to

$$\rho\int_0^h u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U_\infty\frac{dU_\infty}{dx}dy = \tau|_{y=0}^h. \tag{7}$$

We can replace v in the integral via Eq. (2) after noting $v(x, 0) = 0$. This leads to a double integral term; changing the order of integration and integrating once gives

$$\rho\int_0^h 2u\frac{\partial u}{\partial x} - U_\infty\frac{\partial u}{\partial x} - U_\infty\frac{dU_\infty}{dx}dy = -\tau_0, \tag{8}$$

where τ_0 is the shear stress in the fluid at $y = 0$. In the simplest case U_∞ is constant and the integral is zero everywhere outside the boundary layer (since then $u = U_\infty$) and we may replace the upper limit of the integral by the unknown boundary layer thickness $\delta = \delta(x)$ to find

$$\rho\int_0^\delta \frac{\partial}{\partial x}[u(U_\infty - u)]dy = \rho\frac{d}{dx}\int_0^\delta u(U_\infty - u)dy = \tau_0. \tag{9}$$

This is known as the Integral Momentum Equation (IME). In this form it holds for both laminar and turbulent flow and no assumption has been made about the nature of the fluid [7]. However, from now on we will assume that the power law relation, Eq. (5), holds. A similar analysis on (6) leads to the Integral Energy Equation (IEE)

$$\frac{d}{dx}\int_0^{\delta_T} u(T_\infty - T)dy = \kappa\left.\frac{\partial T}{\partial y}\right|_{y=0}, \tag{10}$$

where the thickness of the thermal boundary layer $\delta_T(x) \neq \delta(x)$.

In Section 4, when we develop the approximation method, we will work with derivative forms of these equations and so denote $G = \rho u(U_\infty - u)$, $F = u(T_\infty - T)$. Then we will use derivative forms of the integral equations

$$\text{(a)}\quad \frac{\partial G}{\partial x} = -\frac{\partial \tau}{\partial y} \quad \text{(b)}\quad \frac{\partial F}{\partial x} = -\kappa\frac{\partial^2 T}{\partial y^2}. \tag{11}$$

Note, these equations follow from (9,10) by integrating over the boundary layer. For example, with (11b) we note that $F(\delta_T) = T_y(\delta_T) = 0$ and so

$$\int_0^{\delta_T(x)} \frac{\partial F}{\partial x}dy = \frac{d}{dx}\int_0^{\delta_T(x)} F\,dy \qquad \int_0^{\delta_T(x)} \frac{\partial^2 T}{\partial y^2}dy = -\left.\frac{\partial T}{\partial y}\right|_{y=0}. \tag{12}$$

Eq. (10) then follows immediately.

2.1. Non-dimensionalisation

Using the standard boundary layer scaling for a power law fluid we set

$$u = U_\infty\hat{u}, \quad v = \frac{U_\infty}{Re^{1/(n+1)}}\hat{v}, \quad x = L\hat{x}, \tag{13}$$

$$y = H\hat{y} = \frac{L}{Re^{1/(n+1)}}\hat{y}, \quad \hat{T} = \frac{T - T_s}{T_\infty - T_s}, \tag{14}$$

where U_∞ is assumed constant, the Reynolds number $Re = \rho U_\infty^{2-n} L^n/m$ and L is the plate length. Eq. (1) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right), \tag{15}$$

where the hats have been dropped. The thermal problem is described by