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Porous convection with Cattaneo heat flux

B. Straughan

Department of Mathematical Sciences, Durham University, DH1 3LE, UK

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ABSTRACT

We study the problem of thermal convection in a horizontal layer of Darcy porous material saturated with an incompressible Newtonian fluid, with gravity acting downward. The constitutive equation for the heat flux is taken to be one of Cattaneo type. Care must be taken with the choice of objective derivative for the rate of change of the heat flux. Here we employ a recent model due to Professor C. Christov as well as one suggested many years ago by Professor N. Fox. The thermal relaxation effect in both classes of heat flux law is found to be significant if the Cattaneo number is sufficiently large, and the convection mechanism switches from stationary convection to oscillatory convection with narrower cells. The transition point is calculated and the convection thresholds are derived analytically.

1. Introduction

Heat propagation via a wave mechanism instead of simply by diffusion is one of great current interest. Recent studies confirm this is not simply a low temperature phenomenon, but one which has potentially important real applications. Mundane applications of hyperbolic heat propagation are in fields such as skin burns, Dai et al. [6], chemotaxis, Dolak and Hillen [8], virus spread, Barbera et al. [1], heat transfer in one of Saturn's moons, Bargmann et al. [2], traffic flow, Jordan [11], heat propagation in biological tissues, Vedavarz et al. [31], Mitra et al. [16], phase changes, Liu et al. [14], Miranville and Quintanilla [15], food technology, Saidane et al. [25], and in nanofluids, Vadasz et al. [30]. Other related applications are discussed in Quintanilla and Racke [22,23], Jordan [12], Reverberi et al. [24], Straughan [28], chapters 7 and 8, Vadasz [29], Vadasz et al. [30]. A key way of introducing finite temperature wave motion has been to use the Cattaneo [3] law for the heat flux. In thermal convection in a viscous fluid analysis of the Cattaneo law was initiated by Straughan and Franchi [26], with further work by Lebon and Cloot [13]. The higher derivative Guyer–Krumhansl effects were analysed by Franchi and Straughan [10] and by Dauby et al. [7]. Further work in the area of fluid mechanics employing the Cattaneo law for the heat flux may be found in the interesting papers of Puri and Jordan [19,18], and Puri and Kythe [20,21]. Vadasz [29] investigates whether oscillatory heat motion will be possible in a block of porous material by employing a dual phase lag theory for heat conduction. We here continue with such a study by investigating whether oscillatory motion is possible in a layer of porous material saturated with a viscous fluid which is heated from below.

The heat flux is a vector field and so the equation governing its evolutionary behaviour must involve an objective time derivative. Straughan and Franchi [26] employed an objective time derivative due to *N.* Fox and worked with what might be described as a Cattaneo–Fox theory. Recently, Christov [5] has written a very interesting paper which revisits the question of which objective derivative one should employ when dealing with a Cattaneo type theory for a fluid. He proposes an alternative frame-indifferent generalization of Fourier's law with relaxation of the heat flux. The objective of the current paper is to investigate thermal convection in a layer of saturated porous material employing Cattaneo-like heat flux laws. We employ both the Cattaneo–Fox and Cattaneo–Christov models and compare the results.

2. Equations for porous convection

The equations for thermal convection in a porous medium may be found in the books by Nield and Bejan [17], or by Straughan [27,28]. They consist of the balances of linear momentum, mass, and energy, and these may be written as

$$v_{i,t} = -\frac{1}{\rho}p_{,i} + \alpha g k_i T - \frac{\mu}{\rho K}v_i, \tag{1}$$

$$v_{i,i} = 0, \tag{2}$$

$$\frac{1}{M}T_t + v_i T_{,i} = -Q_{i,i}, \tag{3}$$

where v_i, p, T are the velocity, pressure and temperature fields, $\rho,\ \alpha,\ g,\ \mu$ and K are density, thermal expansion coefficient, gravity,

E-mail address: brian.straughan@durham.ac.uk

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dynamic viscosity and permeability, respectively. The quantity Q_i is the heat flux vector, $\mathbf{k} = (0, 0, 1)$ and standard indicial notation is used throughout. The parameter M is defined below where we derive equation (3).

We now write energy balances for the solid and fluid parts of the porous medium separately, and additionally write a Cattaneo heat flux law for the solid and a Cattaneo–Fox heat flux law for the fluid. This is analogous to the development of the non-Cattaneo case by e.g. [28, pp. 14–15]. Let V_i be the actual velocity of the fluid in the pores, let ϕ be the porosity, and set $v_i = \phi V_i$, v_i being the pore averaged velocity. Denote by subscript s, f the solid and fluid parts. Then we have the energy balance and Cattaneo law for the solid,

$$(\rho_0 c)_s T_{,t} = -\widetilde{Q}_{i,i}, \tag{4}$$

$$\tau_s \widetilde{Q}_{i,t} = -\widetilde{Q}_i - k_s T_{,i}, \tag{5}$$

where c is the specific heat, τ_s is the relaxation time, k_s is the thermal conductivity, and $\widetilde{Q}_i$ is the heat flux. The energy balance in the fluid and the Cattaneo–Fox heat flux laws may be written, cf. Straughan and Franchi [26],

$$(\rho_0 c_p)_f (T_{,t} + V_i T_{,i}) = -\widetilde{Q}_{i,i}, \tag{6}$$

$$\tau_f \left(\widetilde{Q}_{i,t} + V_j \widetilde{Q}_{i,j} - \frac{1}{2} \widetilde{Q}_j V_{i,j} + \frac{1}{2} \widetilde{Q}_j V_{j,i} \right) = -\widetilde{Q}_i - k_f T_{,i}, \tag{7}$$

where c_p is the specific heat at constant pressure, k_f is the thermal conductivity, and τ_f is a relaxation time.

To derive an averaged equation governing the (averaged) porous media properties, we multiply equation (4) by $(1-\phi)$, Eq. (6) by ϕ, and add, and likewise multiply equation (5) by $(1-\phi)$ and Eq. (7) by ϕ, and add. We now define the quantities $Q_i, (\rho_0 c)_m, k_m, \kappa, M$ and τ by

$$Q_i = \frac{\widetilde{Q}_i}{(\rho_0 c_p)_f}, \quad (\rho_0 c)_m = \phi(\rho_0 c_p)_f + (1-\phi)(\rho_0 c)_s,$$
$$k_m = \phi k_f + (1-\phi) k_s,$$
$$\kappa = \frac{k_m}{(\rho_0 c_p)_f}, \quad M = \frac{(\rho_0 c_p)_f}{(\rho_0 c)_m}, \quad \tau = \phi \tau_f + (1-\phi)\tau_s.$$

Then one shows Eqs. (4)–(7) lead to Eq. (3) and

$$\tau Q_{i,t} + \tau_f \left(v_j Q_{i,j} - \frac{1}{2} Q_j v_{i,j} + \frac{1}{2} Q_j v_{j,i} \right) = -Q_i - \kappa T_{,i}. \tag{8}$$

An equivalent derivation to that of Eq. (8), but for the Cattaneo–Christov heat flux law, Christov [5], yields

$$\tau Q_{i,t} + \tau_f (v_j Q_{i,j} - Q_j v_{i,j}) = -q_i - \kappa T_{,i}. \tag{9}$$

The complete system of equations for movement and heat propagation in the fluid in a porous medium therefore consist of (1)–(3) coupled with either (8) or (9).

The saturated porous medium occupies the horizontal layer $\{(x, y) \in \mathbb{R}^2, z \in (0, d)\}$ and Eqs. (1)–(3) with (8) or (9) hold in the domain $\mathbb{R}^2 \times (0, d) \times \{t > 0\}$. The boundary conditions are

$$\begin{aligned} &w \equiv v_3 = 0 \quad \text{on} \quad z = 0, d, \\ &T = T_L, \quad z = 0, \quad T = T_U, \quad z = d, \end{aligned} \tag{10}$$

with $T_L > T_U$, both constants. The steady solution to (1)–(3) with either (8) or (9) which satisfies the boundary conditions (10) and whose stability we are interested in is

$$\bar{v}_i \equiv 0, \quad \overline{T} = -\beta z + T_L, \quad \bar{\mathbf{Q}} = (0, 0, \kappa\beta), \tag{11}$$

where β is the temperature gradient given by

$$\beta = \frac{T_L - T_U}{d}.$$

To analyse the instability of solution (11) we introduce perturbations (u_i, θ, π, q_i) such that $v_i = \bar{v}_i + u_i$, $T = \overline{T} + \theta$, $p = \bar{p} + \pi$, $Q_i = \overline{Q}_i + q_i$. Then, we linearize and non-dimensionalize with the scalings $x_i = x_i^* d$, $t = t^* K\rho/\mu$, $\pi = \pi^* d\mu U/K$, $Pr = \mu/\kappa\rho$, $q_i = q_i^* Q^\sharp$, $Q^\sharp = \kappa T^\sharp/d$, $T^\sharp = U\sqrt{\beta d^2 \mu/\kappa\rho K\alpha g}$, $Da = K/d^2$, $\hat{\tau} = \tau_f/\tau$, where U is a velocity scale. The non-dimensional numbers Pr and Da are the Prandtl and Darcy numbers. Key in this work are the Cattaneo number, C, and Rayleigh number, $Ra = R^2$, introduced as

$$C = \frac{\tau\kappa}{2d^2}, \quad R = \sqrt{\frac{\alpha g d^2 \beta K \rho}{\mu\kappa}}.$$

The linearized, non-dimensional equations which follow from (1)–(3) with (8) are

$$\begin{aligned} &u_{i,t} = -\pi_{,i} + Rk_i\theta - u_i, \\ &u_{i,i} = 0, \\ &\frac{Pr}{MDa}\theta_t = Rw - q_{i,i}, \\ &2C\frac{Pr}{Da} q_{i,t} = CR\hat{\tau}(u_{i,z} - w_{,i}) - q_i - \theta_{,i}, \end{aligned} \tag{12}$$

whereas from (1)–(3) with (9) we derive

$$\begin{aligned} &u_{i,t} = -\pi_{,i} + Rk_i\theta - u_i, \\ &u_{i,i} = 0, \\ &\frac{Pr}{MDa}\theta_t = Rw - q_{i,i}, \\ &2C\frac{Pr}{Da} q_{i,t} = 2CR\hat{\tau} u_{i,z} - q_i - \theta_{,i}. \end{aligned} \tag{13}$$

To study linear instability of the conduction solution (11) we write the variables u_i, θ, q_i and π by explicitly separating the time dependent parts as

$$u_i(\mathbf{x}, t) = e^{\sigma t} u_i(\mathbf{x}), \quad \theta(\mathbf{x}, t) = e^{\sigma t}\theta(\mathbf{x}),$$
$$q_i(\mathbf{x}, t) = e^{\sigma t} q_i(\mathbf{x}), \quad \pi(\mathbf{x}, t) e^{\sigma t}\pi(\mathbf{x}).$$

Then π is eliminated and we put $Q = q_{i,i}$ to reduce (12) or (13) to studying the system

$$\begin{aligned} &\sigma\Delta w = R\Delta^*\theta - \Delta w, \\ &\sigma\frac{Pr}{MDa}\theta = Rw - Q, \\ &2\sigma C\frac{Pr}{Da} Q = -\lambda CR\hat{\tau}\Delta w - Q - \Delta\theta, \end{aligned} \tag{14}$$

where $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian, and where $\lambda = 0$ for the Cattaneo–Christov theory, whereas $\lambda = 1$ when the Cattaneo–Fox model is employed.

The boundary conditions to be used in conjunction with (14) are

$$w = 0, \quad \theta = 0, \quad z = 0, 1, \tag{15}$$

and w and θ satisfy a plane tiling periodicity in the horizontal variables x and y.

3. Linear instability, Cattaneo–Fox theory

In this section, we consider equations (14) when $\lambda = 1$. In the first instance we consider stationary convection, i.e. when $\sigma = 0$. Then one finds from (14),

$$\begin{aligned} &R\Delta^*\theta - \Delta w = 0, \\ &Rw = Q, \\ &\hat{\tau} CR\Delta w = -Q - \Delta\theta. \end{aligned}$$