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Technical Note

Stratified Poiseuille gas flows in horizontal channels

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Abstract

The present short communication shows an exact solution of the Navier–Stokes equations in the case of a channel filled with gas and with temperature contrast between the boundaries. This exact solution is then compared with the result of a numerical simulation made using a numerical code widely used in fire safety engineering. It shows the ability of the code to reproduce this highly stratified flow. Nusselt numbers are then estimated.

Keywords: Navier–Stokes; Exact solution; Stratification; Poiseuille; Channel; Non-Boussinesq

1. Introduction

There are very few exact solutions of the Navier–Stokes equations ([1,8]). The real interest of such exact solutions may be questioned, because they are generally subject to instabilities, sometimes at a relatively low Reynolds number. Moreover, they rarely represent a situation with a practical interest.

However, these exact solutions are useful for making a comparison with a numerical calculation, thus providing some insight on the quality of the simulation tool used.

In the present short communication, we first show that there is an exact solution for gas flows in horizontal channels (of height h) with vertical density gradients which may be very high (leading to a non-Boussinesq situation). The vertical density gradient is created by imposed temperatures on the top and bottom boundaries (see Fig. 1). The flow is considered far downstream of the inlet (say, at a distance L from the entrance, with $L \gg h$, see Fig. 1), so that the precise form of the entrance conditions is unimportant and the gradients in velocity and temperature are purely vertical. This situation is rather different from the one studied in [4], where the flow develops in a gap between two parallel vertical boundaries, gravity being parallel to the main direction of the flow and the small mass flow through each cross section of the gap being small.

In view of the very high density gradients considered, we focus on flows of an ideal gas. The exact solution is then compared with the result of a direct numerical simulation made with a code of widespread use for the assessment of hot air motion with the application of fire safety.

2. Steady laminar parallel flow

2.1. Channel flow equations for an ideal gas at low Mach number

The flow of a fluid of variable density in a 2D channel of height H is considered. The flow is assumed to be parallel and the streamwise direction is denoted by x. The vertical direction is denoted by z, so that

$$\underline{u} = U(z)\underline{e}_x, \quad \rho = \rho(z). \tag{1}$$

The fluid is assumed to be an ideal gas in the low Mach-number limit.

For a fluid of variable density, the Navier–Stokes equations are written as

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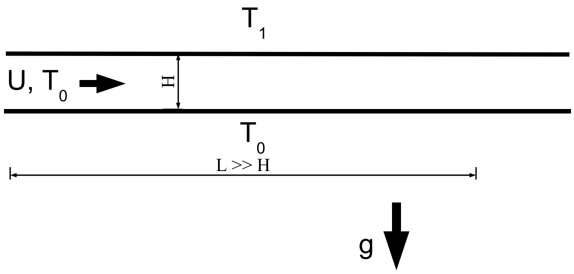

Fig. 1. Stratified flow in a horizontal channel, sketch of the flow and boundary conditions.

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}+g_i$$
$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u_i}{\partial x_i}=0 \tag{2}$$

where τ_{ij} represents the viscous stress and for a Newtonian fluid is given by

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\mu'\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{3}$$

(μ is the dynamic viscosity and $\mu'\approx\mu$). The energy conservation may be expressed through the enthalpy equation, which, for an ideal gas, is written as

$$\frac{\partial \rho C_pT}{\partial t}+\frac{\partial \rho C_pTu_j}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\rho C_p\kappa\frac{\partial T}{\partial x_j}\right) \tag{4}$$

where T is the temperature, C_p is the specific heat and κ is the diffusivity of heat.

Following an infinitesimal stream-tube, $\mathrm{d}P\approx-\rho\mathrm{d}(u^2/2)\approx\rho(u^2/2)$, and therefore, since in an ideal gas the sound celerity is given by $c=\sqrt{\gamma P/\rho}$, with $\gamma=C_p/C_v$

$$\frac{\mathrm{d}P}{P}\approx\frac{\gamma}{2}\mathrm{d}\left(\frac{u^2}{c^2}\right)\approx\frac{\gamma}{2}M^2 \tag{5}$$

where M is the Mach number and $\approx$ reads 'of the same order of magitude'. If $M^2\ll 1$, then $\mathrm{d}P/P\ll 1$. The differential form of the state equation of an ideal gas in the present low Mach number situation is therefore

$$\frac{\mathrm{d}\rho}{\rho}+\frac{\mathrm{d}T}{T}\approx 0, \tag{6}$$

whence

$$\rho T=\rho_0T_0 \tag{7}$$

with ρ_0 the density at a reference temperature T_0.

Therefore, the enthalpy equation reduces to

$$\rho_0C_pT_0\frac{\partial u_j}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\rho C_p\kappa\frac{\partial T}{\partial x_j}\right) \tag{8}$$

where C_p has been taken as a constant.

2.2. Exact solution of the equations

For a steady laminar parallel flow, the equations (2) reduce to

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}z}\left(\mu\frac{\mathrm{d}U}{\mathrm{d}z}\right)$$
$$0=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g$$
$$0=\frac{\mathrm{d}}{\mathrm{d}x_j}\left(\rho C_p\kappa\frac{\mathrm{d}T}{\mathrm{d}x_j}\right)$$

with the boundary conditions as described in the introduction and sketched in Fig. 1.

Defining $a=-\frac{\partial P}{\partial x}$, it follows

$$\frac{\mathrm{d}a}{\mathrm{d}z}=-\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z}\right)=-\frac{\partial}{\partial x}(-\rho g)=0. \tag{9}$$

Hence, the momentum equation on the x-coordinate becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\mu\frac{\mathrm{d}U}{\mathrm{d}z}\right)s=-a, \tag{10}$$

which can be integrated to

$$\mu\frac{\mathrm{d}U}{\mathrm{d}z}=-az+\tau_0 \tag{11}$$

where τ_0 is the shear stress at $z=0$. Note that if $a\neq 0$, $e=\tau/a$ is a height at which $\frac{\mathrm{d}U}{\mathrm{d}z}=0$ ($0<e<H$ exists since $U(0)=U(H)=0$). A good approximation for dry air is a constant Prandtl number $Pr=\frac{\mu}{\rho\kappa}$ and specific heat C_p (see [6], table 5-1-8). Therefore, the heat equation can be integrated to

$$\mu\frac{\mathrm{d}T}{\mathrm{d}z}=\frac{Prq_0}{C_p} \tag{12}$$

where q_0 is the vertical heat flux at $z=0$.

The variation of μ with temperature in an ideal gas is given by the Sutherland formula:

$$\mu=\mu_\mathrm{r}\sqrt{\frac{T}{T_\mathrm{r}}}\frac{1+\frac{C}{T_\mathrm{r}}}{1+\frac{C}{T}} \tag{13}$$

with μ_r the dynamic viscosity at a given reference temperature T_r and C a constant. For dry air, $C=123.6$ K, and at $T_\mathrm{r}=273$ K, $\mu_\mathrm{r}=17.1\times10^{-6}$ Pa.s. Note that the reference temperature may be chosen arbitrary provided the reference dynamic viscosity is calculated according to this reference temperature. It follows that

$$\frac{\mathrm{d}U}{\mathrm{d}z}=\frac{-az+\tau}{\mu_0}\left[\sqrt{\frac{T}{T_0}}\frac{1+\frac{C}{T_0}}{1+\frac{C}{T}}\right]^{-1} \tag{14}$$

$$\frac{\mathrm{d}T}{\mathrm{d}z}=\frac{Prq_0}{C_p\mu_0}\left[\sqrt{\frac{T}{T_0}}\frac{1+\frac{C}{T_0}}{1+\frac{C}{T}}\right]^{-1}. \tag{15}$$

Writing $Z=z/H$, $U_0=\tau H/\mu_0$, $\tilde{u}=U/U_0$, $\alpha=aH/\tau$, $\theta=T/T_0$ (where T_0 is the bottom boundary temperature), $\beta=C/T_0$, and $\gamma=\frac{Prq_0H}{C_p\mu_0T_0(1+\beta)}$, it follows