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Exact factorization technique for numerical simulations of incompressible Navier–Stokes flows

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Abstract

Splitting techniques break an ill-conditioned indefinite system resulting from incompressible Navier–Stokes equations into well-conditioned subsystems, which can be solved reliably and efficiently. Apart from the ambiguity regarding numerical boundary conditions for the pressure (and for intermediate velocities, whenever introduced), splitting techniques usually incur splitting errors which reduce time accuracy. The discrete approach of approximate factorization techniques eliminates the need of numerical boundary conditions and restores time accuracy by an approximate inversion of some matrix in the case of semi-implicit time schemes. For linear implicit, non-linear implicit, and higher-order semi-implicit time schemes, however, approximate factorization techniques are laborious. In this paper, we systematically present a new and straightforward exact factorization technique. The main contributions of this work include: (1) the idea of removing the splitting error or the idea of restoring time accuracy for fully discrete systems, (2) the introduction of the pressure-update type and the pressure-correction type of exact factorization techniques for any time schemes, and (3) an analysis of several established techniques and their relations to the exact factorization technique. The exact factorization technique is implemented with a standard second-order finite volume method and is verified numerically.

Keywords: Exact factorization; Splitting; Numerical boundary conditions; Incompressible Navier–Stokes flows

1. Introduction

The incompressible Navier–Stokes equations in non-dimensional form are described as

$$\frac{\partial u_i}{\partial t} + H_i = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}u_i + f_i, \tag{1}$$

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{2}$$

where $H_i \equiv \frac{\partial}{\partial x_j}(u_j u_i)$, u_i is the velocity component in the x_i direction, f_i is the body force, and Re is the Reynolds number. For decades, a numerical simulation of this system has remained as one of the most interesting topics, and the pressure term is believed by many to be a source for trouble. The indefinite system due to the mixed formulation, in which velocities and pressure are solved simultaneously without any manipulations, is ill-conditioned [9]. In such a system, a relatively small change in some entry of the matrix results in a relatively large change in the solution. Hence, accumulated computer round-off errors or some inherent perturbations of iterative processes make the convergence very hard to achieve. When the size of the system increases or when the physical solutions tend to be more rugged as a consequence of higher Reynolds numbers or discontinuities, the conditioning of the discrete system further deteriorates so that the convergence becomes even more difficult. In some situations, eventually the discrete system becomes singular and no solution can be found. This is why successful simulations of high Re incompressible flows with finite difference methods and finite element

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methods in mixed formulation are rare. In some situations, pressure stabilized methods help to improve the behavior of the discrete system to some degree. However, these methods often invoke some ad hoc parameters and are too expensive for implementation. Also, none of the stabilized methods could reach high-order spatial accuracy, and in fact frequently only a first-order accuracy could be attained. Furthermore, all stabilized methods fail to decrease the size of the discrete system.

To tackle the pressure term, various artificial compressibility (AC) types of methods were invented, such as Chorin's AC for steady flows [5], the consistent penalty method [2], the generalized AC (the iterative Uzawa algorithm) for transient flows [4], and the reduced integration penalty method [18]. Generally speaking, these methods converge slowly and sometimes fail to converge. Often, the range of appropriate parameters to maintain both reliability and efficiency is narrow. Moreover, some methods in this category are not able to produce accurate results for pressure and some are not able to reduce the size of the discrete system. Vorticity-stream function formulation [12] is very competitive in 2-D and 3-D axisymmetric calculations. In general 3-D calculations the stream function does exist [26]; however, both the vorticity and the stream function have three components. Even in 2-D and 3-D axisymmetric calculations, the derived boundary conditions for the vorticity incurs either loss of accuracy or loss of flexibility of numerical methods.

Splitting methods (also known as projection/operator-splitting/time-splitting/fractional-step methods) remain popular among numerical community. The first two papers on the subject were Harlow and Welch's marker-and-cell (MAC) method [17] and Chorin's projection method [6]. Both methods take a first-order explicit scheme, and both require no initial boundary conditions for pressure which is consistent with the original mathematical system. We would like to call them, including Kim and Moin's [21] second-order semi-implicit fractional-step (splitting) method, pressure-update (PU) methods. In PU methods, the pressure or some variable closely related to the pressure is solved according to a Poisson equation. In contrast, one may solve the change of the pressure from a Poisson equation. Examples are second-order methods by van Kan [29] and Bell et al. [1], and higher-order methods by Karniadakis et al. [20]. We would like to refer to the latter category pressure-correction (PC) methods.

The issue of the boundary conditions for the Poisson equation, as well as boundary conditions for intermediate velocities (whenever introduced), has in the past sparked a considerable debate [17,6,7,22,24,8,21,29,15,1,13,14,20,10,23,27,25,3,16] (in chronic order). According to [27], the accuracy of finite difference schemes "depends critically on the boundary condition for the intermediate velocity." However, the numerical boundary condition for the pressure Poisson equation (PPE) is implied in the system already and actually is not required in practice, as shown in the PC type of approximate factorization technique by Dukowicz and Dvinsky [10] and in the PU type of approximate factorization technique by Perot [23]. The exact factorization technique to be introduced in this paper requires no numerical boundary conditions at all.

The elusive issue of splitting error has also drawn sizable attention, in that many of those papers on the issue of numerical boundary conditions also concern the issue of time accuracy. According to Perot [23], a lower-order splitting-induced term in the momentum equation is pointed out as the source of the trouble. Approximate factorization techniques remove the splitting error through an approximate inversion of some matrix. Quarteroni et al. [25] presented a framework for splitting methods and approximate factorization techniques, including Perot's approach. However, the exact factorization technique presented in this paper takes a different path in terms of restoring time accuracy.

In the next section we progressively introduce the exact factorization technique. We start from a specific time and spatial discretization, discuss the approach of approximate factorization to split the system, and introduce the exact factorization technique. Then, we generalize the technique to any time scheme and introduce another version of exact factorization. After that, we compare the technique with the semi-discrete counterpart and make some additional comments on the exact factorization technique. In the following section, a reduced version of exact factorization technique is discussed and comparisons to several well known techniques are made. Implementation and numerical results make up another full section to support the technique.

2. Exact factorization technique

2.1. Temporal and spatial discretizations

The momentum equation (1) indicates that the pressure gradient should stay at the same site with the time derivative of the velocity, in the four-dimensional space–time coordinates system. This implies that at the same time level, the pressure node should stagger from velocity nodes as on the MAC staggered grid. This also implies that the time level for pressure should stagger from the time level for velocity. Since the pressure, which appears only in the momentum equation, is not initially specified, the momentum equation should be displaced from the initial time level.

In light of the above views, we assign time levels for the velocity and for the numerical pressure ϕ as shown in Fig. 1. The idea of numerical pressure was introduced by Kim and Moin [21] and will be further discussed later, but tentatively we may simply regard it as the pressure. The incompressibility is satisfied on time levels for the velocity while momentum equations are satisfied on time levels for pressure. It is noted that in space the incompressibility is satisfied at pressure nodes while momentum equations are satisfied at velocity nodes, just opposite to the