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An improved *r*-factor algorithm for TVD schemes

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Abstract

An improved *r*-factor algorithm for TVD schemes on structured and unstructured grids within a finite volume method framework is proposed for numerical approximation to the convective term. The new algorithm is tested by a problem of pure convection with a double-step profile in an oblique uniform velocity field. The computational results are then compared with the results of Darwish's *r*-factor algorithm using Superbee and Osher limiters on both structured and unstructured grids. The numerical results show that the new algorithm can mitigate the oscillation behavior efficiently while still maintaining the boundedness of the solutions. When using a deferred correction technique to handle the non-linear term arising from the high resolution schemes, the proposed algorithm showed a smoother and faster convergence history on structured grids than Darwish' *r*-factor algorithm, while on unstructured grids the presented one is more accurate with a similar convergence history.

Keywords: Total variation diminishing scheme; Flux limiter; *r*-Factor; Unstructured grids; Deferred correction

1. Introduction

The convective term is seemly simple but hard to deal with in CFD [1]. The difficulties lie in false diffusion, non-conservative, overshoot/undershoot and phase error, etc. [2]. Central schemes work quite well in smooth regions but witness the undesirable severe oscillations around discontinuity. It would seem natural that a numerical scheme should be consistent with the velocity and direction with which information propagates throughout the flow field. Indeed, this is nothing more than obeying the physics of the flow. First-order schemes such as upwinding approach have the advantage that a monotone variation is achieved for the numerical flow-field properties in the vicinity of discontinuities; i.e., no oscillations appear in the numerical solutions around these discontinuities. However, they are diffusive and tend to smear out the flow-field variables, particularly in the vicinity of contact surfaces, which is often unacceptable [3,4]. To mitigate this diffusive effect, some high-order schemes such as second-order upwind schemes (SOU) are developed. Though they work well to diminish the diffusive character of the solution, oscillations which do not exist in the first-order schemes appears [5–7]. Then, to reduce or eliminate this undesirable property, while at the same time retaining the inherent advantages of an upwind scheme, some rather mathematically elegant algorithms have been developed over the past decades. These modern algorithms have introduced such terminology as total variation diminishing (TVD) schemes [8], flux splitting [9], flux limiters [10], Godunov schemes [11], and approximate Riemann solvers [12], etc. These ideas are all broadly classified as upwind schemes since they attempt to properly account for the propagation of information throughout the flow. This paper will discuss only the TVD schemes which are high resolution schemes.

A briefly description of present formulation of *r*-factor using in TVD schemes will be given firstly in the second section, then a new *r*-factor algorithm is proposed based on Darwish' *r*-factor, finally, a test example was illustrate and some conclusions concerning the improved *r*-factor was drawn in the end of the paper.

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Nomenclature

$\boldsymbol{d}$	distance vector
f	face of cell
r	r-factor
N	total number of cells
TV	total variation
x, y, z	components of Cartesian coordinate system
$\boldsymbol{V}$	velocity

Greek symbols

ρ	density of fluid
Φ	independent variable
$\Psi(r)$	flux limiter

Subscripts

i, $i+1$, $i+1/2$	index of cell or face
n, $n+1$	time step
U, C, D, $\mathrm{U_r}$	center of cell

2. The present formulations of r-factor in TVD schemes

Harten [7] introduced the following generalization of Godunov's monotonicity concept [11] in one dimension: if the solution of convection equation changes from time step n to $n+1$ such that

$$\left(\int \left|\frac{\partial \Phi}{\partial x}\right| \mathrm{d}x\right)^{n+1} \leqslant \left(\int \left|\frac{\partial \Phi}{\partial x}\right| \mathrm{d}x\right)^{n} \tag{1}$$

where $\mathrm{TV} = (\int |\frac{\partial \Phi}{\partial x}| \mathrm{d}x)$ was denoted as the total variation of Φ with x, then the scheme is said to be total variation diminishing (TVD).

Eq. (1) can be rewritten in discrete form,

$$\left(\sum_{i=1}^{N-1} |\Phi_{i+1} - \Phi_i|\right)^{n+1} \leqslant \left(\sum_{i=1}^{N-1} |\Phi_{i+1} - \Phi_i|\right)^{n} \tag{2}$$

where Φ_i and Φ_{i+1} denote the x component of the general dependent variable Φ estimated at point (i) and point ($i+1$), N is the number of total cells in computational domain. For a linear scheme, the TVD property is the same as monotonicity. For a non-linear scheme, however, one can maintain the TVD property while achieving higher order (at least in one dimension) by using non-linear functions called limiters to bound the solution variables such that Eq. (2) hold. Since these functions are intended to limit gradients by modifying the flux terms in the difference equations, they are called, quite naturally, flux limiters, which are quite widespread used in modern CFD algorithms [13–16].

The face value $\Phi_{i+1/2}$ of cell (i) in a TVD scheme, on the basis of Roe [17], can be written as the sum of a diffusive first-order upwind and an anti-diffusive term, shown as below:

$$\Phi_{i+1/2} = \Phi_i + \frac{1}{2}\Psi(r_{i+1/2})(\Phi_{i+1} - \Phi_i) \tag{3}$$

The anti-diffusive part is multiplied by the flux limiter function, $\Psi(r)$, which is often a non-linear function of r (also refer as to r-factor), the upwind ratio of consecutive differences of the solution, defined as [15] in structured grids (without loss of generality, assume the velocity at the face $v_{i+1/2} > 0$):

$$r_{i+1/2} = \frac{\Phi_i - \Phi_{i-1}}{\Phi_{i+1} - \Phi_i} \tag{4}$$

For instance, the two limiters [10,18] used in this paper have the forms:

Osher limiter: $\Psi(r) = \max(0, \min(2, r))$.

Superbee limiter:

$$\Psi(r) = \max(0, \min(1, 2r), \min(2, r))$$

However, it is not immediately obvious how to express r on an unstructured grid. Since the index-based notation used in structured grids is not suitable for unstructured grids, the more appropriate notation, shown in Fig. 1 as an example of two-dimensional unstructured grid is adopted. Nodes C and D are defined as the upwind and downwind nodes around face f of cell C, and the virtual node U is defined as the node of upwind of the node C.

Using this notation, Eq. (3) can be rewritten as

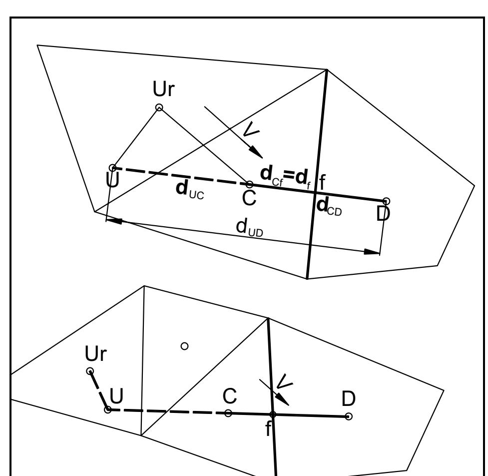

Fig. 1. Advection node stencil.