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Technical Note

Stationary convection–diffusion between two co-axial cylinders

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Abstract

In this note, we examine the high Péclet number limit of the stationary extended Graetz problem for which two families of real and imaginary eigenvalues are associated, respectively, with a downstream convective relaxation and the upstream diffusive establishment. The asymptotic behavior of both families of eigenvalues is studied, in the limit of large Péclet number and thin wall, which bring to the fore a single parameter dependence, previously mentioned in the literature from numerical investigations [M.A. Cotton, J.D. Jackson, in: R.W. Lewis, K. Morgan (Eds.), Numerical Methods in Thermal Problems, vol. IV, Pineridge Press, Swansea, 1985, pp. 504–515]. The fully developed region is specifically studied thanks to the first eigenvalue dependence on the Péclet number, on the thermal conductivity coefficients and on the diameter ratio of the cylinders. The effective transport between the fluid and the solid is investigated through the evaluation of the fully developed Nusselt number and experimental measurements.

Keywords: Micro-systems; Heat transfer; Spectral analysis; Asymptotic

1. Introduction

We hereby analyze some special limit of the extended Graetz problem using variable separation eigenfunctions. The mathematical and numerical solutions for this problem has been obtained in numerous previous contribution [5,6,11,10,9]. From the precursory contributions of Papoutsakis et al. [5,6], a complete representation of the solution relies on an orthogonal eigenfunction expansion of this problem, independently from the applied boundary conditions for the external cylinder, the input and the output conditions. The core of any explicit numerical computation of the temperature field relies on the evaluation of a subset of the infinite discrete spectrum of eigenvalues, and eigenfunctions.

In this note we focus on some simple expression for the asymptotic behavior of the solution, which put forward a simple parameter already heuristically proposed in previous contribution. The effective transfer between the tube and the co-axial solid cylinder in the fully developed region is studied through the computation of the Nusselt number in Section 4. A comparison with the available experimental results is discussed in the last section.

2. General solution and eigenvalue problem

The extended Graetz problem is considered for two complementary configurations sketched in Fig. 1 that we will subsequently refer to as a and b. The following non-dimensional variables are introduced to describe the problem:

$$\eta = \frac{r}{r_a}, \quad \zeta = \frac{z}{r_a}, \quad R = \frac{r_b}{r_a}, \quad Pe = \frac{2Ur_a}{D^{\mathrm{I}}},$$

where r_b is the radius of the solid co-axial cylinder. A fully developed hydrodynamic flow inside the tube is considered. The velocity longitudinal component u along the z-axis of the tube has a Poiseuille parabolic profile which is proportional to the mean applied pressure gradient $\partial_z P$

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Nomenclature

D^{I}, D^{II}	thermal diffusivity in the fluid (I) and in the solid (II)	P	fluid pressure
F_R	eigenfunction in the solid cylinder	S	surface of the inner pipe
G	Graetz eigenfunction	T	temperature
I	dimensionless asymptotic parameter	T_0	reference temperature at infinity
J_0, J_1	zeroth and first Bessel functions of the first kind	$\Theta = T - T_0$	intrinsic temperature
k^{I}, k^{II}	thermal conductivity coefficients	U	averaged longitudinal velocity
Y_0, Y_1	zeroth and first Bessel functions of the second kind	u	fluid longitudinal velocity
Nu	Nusselt number	z	longitudinal coordinate
S	surface of the cylinder		
r_a, r_b	internal radius of the inner and outer cylinder	*Greek symbols*	
Pe	Péclet number	$\eta = \frac{r}{r_a}$	dimensionless radial coordinate
r	radial coordinate	λ	eigenvalue of the coupled thermic problem
$R = \frac{r_b}{r_a}$	aspect ratio between the inner and the outer cylinder	$\Phi(a,b,z)$	confluent hyper-geometric function
		$\zeta = \frac{z}{r_a}$	dimensionless longitudinal coordinate

$$u = \frac{1}{4}\partial_z P(r_a^2 - r^2) = \frac{2U}{r_a^2}(r_a^2 - r^2),$$
$$U = \frac{1}{S}\int_S u\,\mathrm{d}S = \frac{2}{r_a^2}\int_0^{r_a} ur\,\mathrm{d}r,$$

where U is the averaged fluid velocity inside the tube having section S and radius r_a. Only axi-symmetric boundary conditions will be considered in the following. The non-dimensional Stationary convection–diffusion of heat is described by

$$Pe\,\partial_\zeta\Theta = \frac{1}{\eta(1-\eta^2)}\partial_\eta(\eta\partial_\eta\Theta) \quad \text{in I},$$
$$\Delta\Theta = 0 \quad \text{in II}, \tag{1}$$

where $\Theta = T - T_0$ is a relative temperature built on the reference temperature T_0 in the fluid tube I and the solid region II at $z \to +\infty$ for configuration 1a and at $z \to -\infty$ for configuration 1b, as already used in [5,6]. We restrict our attention to the solution region where there is an adiabatic isolated solid cylinder, i.e. $\zeta > 0$ in configuration 1a, and $\zeta < 0$ in configuration 1b

$$\partial_\eta\Theta^{\mathrm{II}}(R,\zeta) = 0. \tag{2}$$

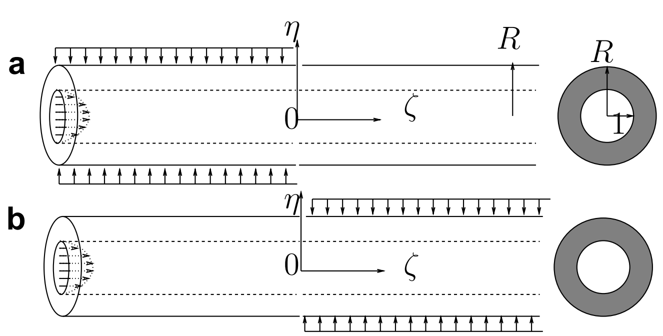

Fig. 1. Schematic representation of the two symmetrical configurations under study.

Firstly we are looking at far field decreasing boundary conditions,

$$\Theta^{\mathrm{I}} \to \Theta^{\mathrm{II}} \to 0 \quad |\zeta| \to \infty. \tag{3}$$

While the temperature and flux equilibrium between the fluid and solid region reads

$$k^{\mathrm{I}}\partial_\eta\Theta^{\mathrm{I}}(1,\zeta) = k^{\mathrm{II}}\partial_\eta\Theta^{\mathrm{II}}(1,\zeta), \tag{4}$$
$$\Theta^{\mathrm{I}}(1,\zeta) = \Theta^{\mathrm{II}}(1,\zeta), \tag{5}$$

where k^{I}, k^{II} are the thermal conductivity in the fluid and in the solid.

One can find a general solution to the problem (1) by writing it as

$$\Theta^{\mathrm{I}}(\eta,\zeta) = \sum_n \theta_{n\pm} G(\lambda_{\pm n},\eta)\mathrm{e}^{-\lambda_{\pm n}^2\zeta/Pe},$$
$$\Theta^{\mathrm{II}}(\eta,\zeta) = \sum_n \theta_{n\pm} F_R\left(\frac{\lambda_{\pm n}^2}{Pe}\eta\right)\mathrm{e}^{-\lambda_{\pm n}^2\zeta/Pe}, \tag{6}$$

where

$$F_R\left(\frac{\lambda_n^2}{Pe}\eta\right) = J_0\left(\frac{\lambda_n^2}{Pe}\eta\right)Y_1\left(\lambda_n^2\frac{R}{Pe}\right) - J_1\left(\lambda_n^2\frac{R}{Pe}\right)Y_0\left(\frac{\lambda_n^2}{Pe}\eta\right)$$

is the linear combinations of the Bessel harmonic eigenfunctions [1] which fulfills the radial adiabatic boundary condition (2).

$G(\lambda,\eta)$ is the Graetz function:

$$G(\lambda,\eta) = \mathrm{e}^{-\lambda\eta^2/2}\Phi\left(\frac{1}{2} - \frac{\lambda}{4}, 1, \lambda\eta^2\right),$$

where $\Phi(a,b,z)$ is the confluent hyper-geometric function [1] (sometimes referred to as ${}_1F_1$) which possesses two a and b parameters and one variable z. Unlike the solution of the Graetz problem, the solutions families (6) explicitly depend on the Péclet number, on the radius ratio of the fluid and solid region and on the conductivity ratio $k^{\mathrm{I}}/k^{\mathrm{II}}$ through condition (4).