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On flux terms in volume averaging

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ABSTRACT

This note examines the modeling of non-convective fluxes (e.g., stress, heat flux and others) as they appear in the general, unclosed form of the volume-averaged equations of multiphase flows. By appealing to the difference between slowly and rapidly varying quantities, it is shown that the natural closure of these terms leads to the use of a single, slowly-varying combined average flux, common to both phases, plus rapidly-varying local contributions for each phase. The result is general and only rests on the hypothesis that the spatial variation of the combined average flux is adequately described by a linear function of position within the averaging volume. No further hypotheses on the nature of the flow (e.g., about specific flow regimes) prove necessary. The result agrees with earlier ones obtained by ensemble averaging, is illustrated with the example of disperse flows and discussed in the light of some earlier and current literature. A very concise derivation of the general averaged balance equation is also given.

1. Introduction

Volume averaging is a standard method for the derivation of averaged equations of balance for the modeling of multiphase flows (see e.g. Nigmatulin et al., 1996; Whitaker, 1999; Prosperetti and Tryggvason, 2009; Ishii and Ibiki, 2011). While the formal application of the method is relatively straightforward, the interpretation of the resulting equations is subtle. This is a crucial point as, in the absence of physical transparency, it becomes harder to develop physically relevant closures.

The specific aspect on which we focus in this note is the modeling of non-convective fluxes (e.g. stress, heat flux, diffusive flux) in the averaged balance equations. The basis of the approach is the recognition that, in a spatially non-homogeneous flow, simultaneously with the slow spatial dependence of the macroscopic averaged quantities, there is a faster, local spatial dependence of the microscopic fields. A procedure which takes into consideration this multi-scale nature of the actual situation results in a physically transparent form of the averaged equations (Section 3) which, in turn, helps to close the equations as we show with the example of a disperse flow in Section 4. Section 5 provides a discussion of the results in the context of the existing literature. Section 2 contains a very synthetic ("efficient") derivation of the volume-averaged equations in the standard form in which they are usually presented.

2. The general averaged balance law

We consider for simplicity a general system consisting of two phases, denoted by indices 1 and 2, although extension of the procedure to the more general case of three or more phases is straightforward.

We attach to each point $\mathbf{x}$ in space an averaging volume $V(\mathbf{x})$ of fixed shape and orientation; $V_j(\mathbf{x}, t)$, with $j = 1$ or 2, denotes the (generally time-dependent) part of V occupied by the jth phase so that $V = V_1 + V_2$. The surface S of V is also decomposed in the same way, $S = S_1 + S_2$, with S_j the portion of S occupied by the jth phase (see Fig. 1). Inside V the two phases are separated by an interface S_i, possibly consisting of disjoint parts as, e.g., in the case of droplets suspended in a continuous phase. It is important to keep in mind that the volume occupied by the jth phase inside the averaging volume is bounded by $S_j + S_i$ which, therefore, is a closed surface.

The volume average of a generic quantity q_j of arbitrary tensorial order belonging to the jth phase is defined as

$$\langle q_j \rangle(\mathbf{x}, t) = \frac{1}{V_j(\mathbf{x}, t)} \int_{V_j(\mathbf{x},t)} q_j(\boldsymbol{\xi}, t) d^3\xi \,, \tag{1}$$

where $\mathbf{x}$ is the position of the averaging volume. Upon using the (generalized) Reynolds transport theorem we have

$$\frac{\partial}{\partial t}\left(V_j \langle q_j \rangle\right) = \int_{V_j} \frac{\partial q_j}{\partial t} d^3\xi + \oint_{S_j+S_i} q_j \mathbf{v}\cdot\mathbf{n}_j dS \,, \tag{2}$$

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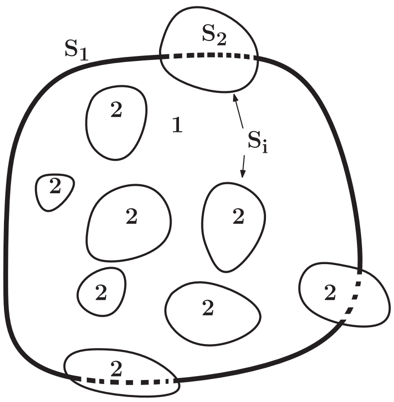

Fig. 1. Averaging volume in a two-phase system. The surface of the averaging volume, $S = S_1 + S_2$, consists of a portion S_1 in contact with phase 1 (continuous thick line) and a portion S_2 in contact with phase 2 (dashed thick line). Inside the averaging volume the two phases are separated by an interface S_i (thin lines) which may consist of disjoint parts as in this figure.

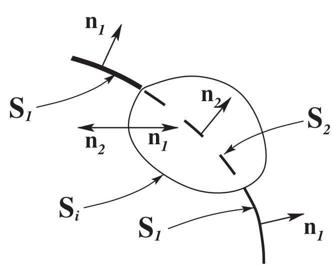

Fig. 2. The unit normals are defined to be directed out of the corresponding phase.

where $\mathbf{v}$ is the local velocity of the integration surface and $\mathbf{n}_j$ is the unit normal directed out of the jth phase (Fig. 2). Since V is fixed, $\mathbf{v} = 0$ on S_j and we are left with

$$\frac{\partial}{\partial t}\big(V_j\langle q_j\rangle\big) = \int_{V_j}\frac{\partial q_j}{\partial t}d^3\xi + \int_{S_i} q_j\mathbf{v}_i\cdot\mathbf{n}_j dS_i\,, \tag{3}$$

where $\mathbf{v}_i$ is the velocity of the interface contained within V. We assume that the quantity q_j satisfies a general balance equation of the form

$$\frac{\partial q_j}{\partial t} = -\nabla\cdot(\mathbf{u}_j q_j) + \nabla\cdot\boldsymbol{\phi}_j + \theta_j\,, \tag{4}$$

where $\mathbf{u}_j$ is the jth phase velocity, $\boldsymbol{\phi}_j$ the non-convective flux of q_j and θ_j the volume source of q_j. Upon substituting into the first term in the right-hand side of (3) we find

$$\frac{\partial}{\partial t}\big(V_j\langle q_j\rangle\big) = \int_{V_j}\big[-\nabla\cdot(\mathbf{u}_j q_j - \boldsymbol{\phi}_j) + \theta_j\big]d^3\xi + \oint_{S_i} q_j\mathbf{v}_i\cdot\mathbf{n}_j dS_i\,, \tag{5}$$

or, upon using the divergence theorem,

$$\frac{\partial}{\partial t}\big(V_j\langle q_j\rangle\big) + \int_{S_j} q_j\mathbf{u}_j\cdot\mathbf{n}_j dS_j = \int_{S_j}\boldsymbol{\phi}_j\cdot\mathbf{n}_j dS_j + \int_{S_i}\big[-q_j(\mathbf{u}_j - \mathbf{v}_i) + \boldsymbol{\phi}_j\big]\cdot\mathbf{n}_j dS_i + \int_{V_j}\theta_j d^3\xi\,. \tag{6}$$

Here we have separated the surface integrals over the interface (in the right-hand side) from those over the surface of the averaging volume (in the left-hand side).

Now we use the exact, purely geometric theorem (see e.g., Gray and Lee, 1977; Prosperetti and Tryggvason, 2009)

$$\int_{S_j}\boldsymbol{\phi}_j\cdot\mathbf{n}_j dS_j = \nabla\cdot\int_{V_j}\boldsymbol{\phi}_j d^3\xi = \nabla\cdot\big(V_j\langle\boldsymbol{\phi}_j\rangle\big)\,, \tag{7}$$

(actually valid for any vector or higher-order tensor) to rewrite this result as

$$\frac{\partial}{\partial t}\big(V_j\langle q_j\rangle\big) + \nabla\cdot\big(V_j\langle q_j\mathbf{u}_j\rangle\big) = \nabla\cdot\big(V_j\langle\boldsymbol{\phi}_j\rangle\big) + \int_{S_i}\big[-q_j(\mathbf{u}_j - \mathbf{v}_i) + \boldsymbol{\phi}_j\big]\cdot\mathbf{n}_j dS_i + V_j\langle\theta_j\rangle\,. \tag{8}$$

Upon division by V, assumed to be independent of $\mathbf{x}$ as already stated, and upon introduction of the volume fraction α_j of the j-phase defined by

$$\alpha_j = \frac{V_j}{V}\,, \tag{9}$$

we find the general averaged balance law

$$\frac{\partial}{\partial t}\big(\alpha_j\langle q_j\rangle\big) + \nabla\cdot\big(\alpha_j\langle q_j\mathbf{u}_j\rangle\big) = \nabla\cdot\big(\alpha_j\langle\boldsymbol{\phi}_j\rangle\big) + \frac{1}{V}\int_{S_i}\big[-q_j(\mathbf{u}_j - \mathbf{v}_i) + \boldsymbol{\phi}_j\big]\cdot\mathbf{n}_j dS_i + \alpha_j\langle\theta_j\rangle\,. \tag{10}$$

3. Separation of scales

While (10) is formally exact, its physical transparency can be enhanced as we now show. We focus on the terms containing the non-convective flux $\boldsymbol{\phi}_j$ in (8), namely

$$\boldsymbol{\Phi}_j \equiv \nabla\cdot\big(V_j\langle\boldsymbol{\phi}_j\rangle\big) + \int_{S_i}\boldsymbol{\phi}_j\cdot\mathbf{n}_j dS_i\,. \tag{11}$$

We show the development explicitly for $j = 1$; the results for $j = 2$ can be recovered by simply interchanging the indices 1 and 2.

Let us define

$$\overline{\boldsymbol{\phi}} = \alpha_1\langle\boldsymbol{\phi}_1\rangle + \alpha_2\langle\boldsymbol{\phi}_2\rangle\,. \tag{12}$$

By its definition, this quantity captures the large-scale structure of the $\boldsymbol{\phi}$ field and may be expected to be slowly varying over the scale of the averaging volume. Upon multiplying by V (assumed independent of position) and taking the divergence we have

$$V\nabla\cdot\overline{\phi} = \nabla\cdot(V_1\langle\phi_1\rangle) + \nabla\cdot(V_2\langle\phi_2\rangle)\,. \tag{13}$$

This relation permits us to re-write (11) with $j = 1$ as

$$\boldsymbol{\Phi}_1 = V\nabla\cdot\overline{\boldsymbol{\phi}} - \nabla\cdot(V_2\langle\boldsymbol{\phi}_2\rangle) + \int_{S_i}\boldsymbol{\phi}_1\cdot\mathbf{n}_1 dS_i\,. \tag{14}$$

On the phase interface S_i the conservation law of the generic quantity q_j imposes a relation between $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ of the general form

$$\big(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\big)\cdot\mathbf{n}_1 = \gamma\,, \tag{15}$$

where γ is a surface source term. For example, when $\boldsymbol{\phi}$ is the stress, γ would be a vector accounting for surface tension effects at the interface separating two fluids. For a liquid-vapor system, when $\boldsymbol{\phi}$ is the heat flux, γ would account for the latent heat effects associated to phase change, and so on. Upon using the fact that, on the interface S_i, $\mathbf{n}_1 = -\mathbf{n}_2$ as the normals are defined to be directed out of the corresponding phase (see Fig. 2), we can re-write (14) as

$$\boldsymbol{\Phi}_1 = V\nabla\cdot\overline{\boldsymbol{\phi}} - \nabla\cdot(V_2\langle\boldsymbol{\phi}_2\rangle) - \int_{S_i}\boldsymbol{\phi}_2\cdot\mathbf{n}_2 dS_i + \int_{S_i}\gamma\, dS_i\,, \tag{16}$$

or, by the geometric theorem (7),

$$\boldsymbol{\Phi}_1 = V\nabla\cdot\overline{\boldsymbol{\phi}} - \oint_{S_2+S_i}\boldsymbol{\phi}_2\cdot\mathbf{n}_2 dS + \int_{S_i}\gamma\, dS_i\,. \tag{17}$$

Let us now set in this equation

$$\boldsymbol{\phi}_2 = \overline{\boldsymbol{\phi}} + \boldsymbol{\phi}_2'\,. \tag{18}$$