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Finite thermal convection of non-Fourier fluids

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ABSTRACT

The thermal convection is examined theoretically for fluids possessing significant thermal relaxation time, characterizing the response of the heat flux and the temperature gradient to changes in one another. Fourier's law breaks down for such fluids. Non-Fourier heat transport occurs in a wide range of applications, including superfluid helium, fluids subjected to rapid heating, and strongly confined fluids. The parallels between non-Fourier fluids and viscoelastic polymeric solutions are established. For viscoelastic fluids, the constitutive equation for stress must be frame invariant, a condition that must also hold for the constitutive equation for heat flux. The stability of conduction state is first reviewed, and the convection state is obtained using a low-order dynamical system approach. The stability of steady convection is analysed, and the Nusselt number is obtained as function of the Rayleigh and Cattaneo numbers.

1. Introduction

Heat transfer is typically described by Fourier's law, which is given by

$$\mathbf{Q} = -K\nabla T. \tag{1.1}$$

Here, $\mathbf{Q}$ is the heat flux, K is the thermal conductivity of the medium, and T is the temperature. When combined with the First Law of Thermodynamics, Fourier's law predicts an infinite speed of heat propagation. Physically, however, a disturbance in T must travel at a finite speed that is determined by molecular interactions [1]. One approach to remedy this problem has been to add a partial time derivative to the left-hand side of Eq. (1.1), as in the case of the Maxwell–Cattaneo equation [2]. This results in a hyperbolic differential equation, implying wave-like heat transport. This does not necessarily solve the problem of instantaneous heat propagation, however [3–5], since the Maxwell–Cattaneo equation is not frame-invariant and, as such, its application is restricted to non-deformable media. Heat transport equations involving different objective derivatives have been introduced in attempts to remedy this situation. The most promising modification appears to be that of [6], recently revisited by Ref. [7], whose use of Oldroyd's upper-convected derivative [8] leads to the frame indifferent Cattaneo–Vernotte equation,

$$\tau\frac{\delta\mathbf{Q}}{\delta t} + \mathbf{Q} = -K\nabla T \tag{1.2}$$

where

$$\frac{\delta\mathbf{Q}}{\delta t} \equiv \frac{\partial\mathbf{Q}}{\partial t} + V\cdot\nabla\mathbf{Q} - \mathbf{Q}\cdot\nabla V - \mathbf{Q}\nabla\cdot\mathbf{V}. \tag{1.3}$$

τ is the thermal relaxation time of the medium and characterizes the relaxation of the heat flux to a new steady state following a perturbation of the temperature field.

Coupled with the energy equation, this constitutive equation yields a single equation for T(x,t), an advantage that other frame-invariant formulations do not possess [6]. This equation replaces Fourier's law whenever non-Fourier effects are relevant, collapses back to Fourier's law when they are not, and can be applied to both deformable and non-deformable media. In this paper, we refer to fluids in which the effects of τ are non-negligible as non-Fourier fluids, while those in which the relaxation time can be ignored are referred to as Fourier fluids.

Most practical problems involve media with relaxation times on the order of picoseconds [9] and, in such cases, the Cattaneo–Vernotte equation reduces to the classical Fourier model. There are systems, however, in which the relaxation time is not negligible. Non-Fourier effects lead to thermal waves in superfluid

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liquid helium, referred to as second sound [10,11], and have been increasingly observed in a variety of other systems as well. For example at small lengths scales, the heat transport properties of rarefied gases [12,13], and convection around MEMS devices have been explained in terms of non-Fourier behaviour [14–16], and non-Fourier contributions to heat transport have been predicted in theories of granular flows [17,18], as well as nanofluids [19].

The importance of non-Fourier effects is characterized by the ratio of the thermal relaxation time to the time scale for thermal diffusion. If D is a characteristic length scale and κ the thermal diffusivity of the fluid, then the thermal diffusion time is D^2/κ. Non-Fourier effects become significant when the ratio $C = \tau\kappa/D^2$, referred to as the Cattaneo number, becomes significant. C increases relatively rapidly as D decreases, and so non-Fourier effects are expected to be observable in very small systems such as micro- and nanometer devices that involve heat transport and flow [15,20–28].

There has been a limited amount of work done on convection with non-Fourier heat transport [3,29–31]. In this paper, we analytically investigate the linear stability of the steady conduction state in a Rayleigh–Bénard configuration with $C > 0$, as well as the heat transport and stability of the steady convection state that bifurcates from the conduction state.

2. Problem formulation

2.1. Governing equations and boundary conditions

Consider a thin layer of a Newtonian non-Fourier liquid of infinite extent in the (x,y) directions, confined between isothermal plates at $Z = 0$ and $Z = D$. The fluid layer is heated from below, with the plates maintained at temperatures $T_0 + \delta T$ and T_0, respectively. When δT is small, there is no flow and the heat transport across the layer is solely due to conduction. As δT is increased, thermal expansion causes the density of the liquid near the lower plate to decrease. When the decrease in gravitational potential energy, which results from raising the less-dense fluid to the top of the layer, becomes larger than the energy dissipated by viscosity and thermal diffusion, a convective flow develops.

The fluid density ρ is assumed to depend linearly on the temperature T according to

$$\rho = \rho_0[1 - \alpha_T(T - T_0)], \tag{2.1}$$

where α_T is the coefficient of thermal expansion and ρ_0 is the density of the fluid at T_0. The fluid is assumed to be incompressible, with specific heat at constant pressure c_p, thermal conductivity K and viscosity μ. The fluid behaviour is described by equations for the conservation of mass, linear momentum and energy, as well as the constitutive equation for the non-Fourier heat flux. In this case, the conservation equations are given by

$$\nabla\cdot\mathbf{V} = 0, \tag{2.2}$$

$$\rho_0(\mathbf{V}_t + \mathbf{V}\cdot\nabla\mathbf{V}) = -\nabla P - \rho g\hat{z} + \mu\Delta\mathbf{V}, \tag{2.3}$$

$$\rho_0 c_p(T_t + \mathbf{V}\cdot\nabla T) = -\nabla\cdot\mathbf{Q}, \tag{2.4}$$

where Δ is the Laplacian operator and the subscript t denotes partial differentiation with respect to time. Here $\mathbf{V} = (U, 0, W)$ is the velocity vector, P is the pressure, g is the acceleration due to gravity, and $\hat{z}$ is a unit vector in the z-direction. In writing Eqs. (2.2)–(2.4) we have used the Boussinesq approximation, which states that the effect of the variations in density are negligible everywhere in the conservation equations except in the buoyancy term of Eq. (2.3) [32]. We take the heat flux to be governed by the Cattaneo–Vernotte equation introduced above, re-written here as

$$\tau(\mathbf{Q}_t + \mathbf{V}\cdot\nabla\mathbf{Q} - \mathbf{Q}\cdot\nabla\mathbf{V}) = -\mathbf{Q} - K\nabla T. \tag{2.5}$$

We use free–free boundary conditions and perfectly conducting upper and lower plates, such that the boundary conditions are

$$\begin{aligned} &W(X, Z = 0, t) = W(X, Z = D, t) = 0,\\ &W_{zz}(X, Z = 0, t) = W_{zz}(X, Z = D, t) = 0,\\ &T(X, Z = 0, t) = T_0 + \delta T, \quad T(X, Z = D, t) = T_0. \end{aligned} \tag{2.6}$$

While other boundary conditions could be adopted, the free–-free conditions are convenient due to the mathematical simplicity of the corresponding solutions for $\mathbf{V}$ and T [33].

The base state of the system of Eqs. (2.2)–(2.5) with the boundary conditions in Eq. (2.6) corresponds to no flow. Both the transient and upper convective terms in Eq. (2.5) vanish in this state, and transport of heat occurs simply by conduction. Consequently, the temperature, pressure gradient and heat flux in this state are given by

$$\begin{aligned} &T_B = -(Z/D)\delta T + T_0 + \delta T,\\ &dP_B/dZ = -\rho_0[1 - \alpha_T\delta T(1 - Z/D)]g,\\ &\mathbf{Q}_B = \left(0, K\frac{\delta T}{D}\right), \end{aligned} \tag{2.7}$$

respectively, where the subscript B refers to the base state. The problem is conveniently cast in dimensionless form by taking the length, time and velocity scales as D, D^2/κ and κ/D, respectively. Let $p = D^2/\kappa\mu(P - P_B)$, $\mathbf{q} = D/K\delta T(\mathbf{Q} - \mathbf{Q}_B)$ and $\theta = T - T_B/\delta T$ be the dimensionless deviations of the pressure, heat flux and temperature from their values in the base state. Substituting these into Eqs. (2.2)–(2.5) the dimensionless equations for these deviations are

$$\nabla\cdot\mathbf{v} = 0, \tag{2.8}$$

$$Pr^{-1}(\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p + Ra\theta\,\mathbf{e}_z + \Delta\mathbf{v}, \tag{2.9}$$

$$\theta_t + \mathbf{v}\cdot\nabla\theta = -\nabla\cdot\mathbf{q} + w, \tag{2.10}$$

$$C(\mathbf{q}_t - \mathbf{v}_z + \mathbf{v}\cdot\nabla\mathbf{q} - \mathbf{q}\cdot\nabla\mathbf{v}) = -\mathbf{q} - \nabla\theta, \tag{2.11}$$

where $\mathbf{v} = (u, 0, w)$ is the dimensionless velocity vector. There are two linear terms of non-Fourier origin in Eq. (2.11): the transient term proportional to $\mathbf{q}_t$ and the term involving $\mathbf{v}_z$. The non-dimensional Prandtl number, Rayleigh number, and Cattaneo number are given by

$$Pr = \frac{\nu}{\kappa}, \quad Ra = \frac{\delta T\alpha_T g D^3}{\nu\kappa}, \quad C = \frac{\tau\kappa}{D^2}, \tag{2.12}$$

respectively, where $\kappa = K/\rho_0 c_p$ is the thermal diffusivity.

The heat flux can be eliminated from Eqs. (2.10) and (2.11) by taking the divergence of Eq. (2.11), using the identity $\nabla\cdot(\mathbf{a}\cdot\nabla\mathbf{b}) = \nabla\mathbf{a} : \nabla\mathbf{b} + \mathbf{a}\cdot\nabla(\nabla\cdot\mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are two general vectors, and using (2.8) and (2.10). We obtain

$$C[\theta_{tt} + 2\mathbf{v}\cdot\nabla\theta_t + \mathbf{v}_t\cdot\nabla\theta - w_t + \mathbf{v}\cdot\nabla(\mathbf{v}\cdot\nabla\theta)] + \theta_t + \mathbf{v}\cdot\nabla\theta - w = \nabla^2\theta. \tag{2.13}$$

Since the problem is two-dimensional, we introduce the stream function $\psi(x, z, t)$, such that $u = \psi_z$, $w = -\psi_x$. Finally, taking the