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# On a solution of the fuzzy Dirichlet problem for the heat equation

N.A. Gasilov [a], Ş. Emrah Amrahov [b,*], A.G. Fatullayev [a]

[a] Baskent University, Ankara, 06810, Turkey
[b] Computer Engineering Department, Ankara University, Turkey

A R T I C L E I N F O

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A B S T R A C T

In real-world applications, the behavior of the system is determined by physics laws and described by differential equations. In particular, the heat transfer is determined by Fourier's law of heat conduction and described by partial differential equation of parabolic type. When we construct a mathematical model, we need several parameters, a lot of them are obtained from measurements, or observations. In general, no measurement is perfect and these parameters become uncertain. Fuzzy sets are a useful tool to model such uncertainties. Thus, mathematical models arise, where due to physical laws the dynamics is crisp (certain) but some parameters (such as the source term, initial and boundary values) are fuzzy. In this paper, we consider such a model. Namely, we investigate fuzzy Dirichlet problem for the heat equation with fuzzy source function and fuzzy initial-boundary conditions.

Most of the researchers assume a solution of a fuzzy differential equation as a fuzzy-valued function. But this approach is accompanied with some known difficulties. We are motivated by the fact that the fuzzy-valued function is not the only tool to model the uncertainties, changing with time. We are looking for a solution in the form of a fuzzy set (bunch) of real functions. We assume that the source term and initial-boundary conditions are modeled by triangular fuzzy functions, which are a special kind of fuzzy bunches. To determine the solution, we split the given fuzzy Dirichlet problem to three subproblems. The first subproblem provides the crisp solution. The other two subproblems give the uncertainties due to initial-boundary conditions and due to source function. We show that these uncertainties are triangular fuzzy functions. On the basis of the obtained results, we establish the existence and uniqueness theorem for the solution, under commonly accepted conditions.

We propose a solution method and explain it on numerical examples.



## 1. Introduction

Since knowledge about dynamical systems, modeled by differential equations, is often incomplete or vague, many mathematical models of physical, chemical and biological phenomena are described by fuzzy partial differential equations (FPDEs) [16]. The first study on FPDEs belongs to Buckley and Feuring [3]. They consider solutions to elementary FPDEs and apply the same strategy as in Ref. [4] which is first check to see if the proposed method produces a solution and else see if the Seikkala procedure [20] generates a solution. But this approach works only in some simple cases. Jafelice et al. consider an application of PDEs with fuzzy parameters obtained through fuzzy rule-based systems [13]. Chen et al. present a new inference method with applications to FPDEs [5]. Oberguggenberger describes fuzzy and weak solutions for PDEs [17].

Salahshour and Haghi [19] solve the fuzzy heat equation under strongly generalized H-differentiability. Based on the fuzzy Laplace transform, they convert the original fuzzy heat equation to the corresponding two-point fuzzy boundary value problem.

Karami et al. [14] use fuzzy logic to predict the heat transfer in an air cooled heat exchanger equipped with tube inserts of three types (butterfly, classic and jagged twisted tape). The results show that the fuzzy technique has low error rate: the average error was found to be 0.68% as compared with experimental data.

Bertone et al. [2] investigate heat, wave and Poisson equations as classical models of partial differential equations with uncertainty. In each problem only one parameter (diffusion coefficient

* Corresponding author. Tel.: +90 312 2033300x1771.
*E-mail addresses:* gasilov@baskent.edu.tr (N.A. Gasilov), emrah@eng.ankara.edu.tr (Ş.E. Amrahov), afet@baskent.edu.tr (A.G. Fatullayev).

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in heat equation, speed coefficient in wave equation and permittivity coefficient in Poisson equation) is taken to be uncertain, considering it as a fuzzy number. The authors build the fuzzy solution from the deterministic solution using Zadeh's extension principle. They prove the stability of the fuzzy solution with respect to the initial-boundary data, and show that as time goes to zero, the diameter of the uncertainty converges to zero.

Some researchers develop numerical methods for solving FPDEs. Allahviranloo and Kermani [1] use the finite difference method for numerical solution of FPDEs. Mikaeilvand and Khakrangin [15] propose a transform method to solve FPDEs. They use the fuzzy two-dimensional differential transform method of fixed grid size to find approximate solutions. Stepnicka and Valasek [21] apply fuzzy transform technique to find numerical solution of crisp PDEs.

In this study to solve FPDEs we develop the method proposed by Gasilov et al. [7–12]. The main difference from all other studies which investigate FPDEs is that we look for the solution as a fuzzy set (bunch) of real functions, not as a fuzzy-valued function. The novelties of the presented paper are: 1) The proposed method is applied to FPDEs for the first time; 2) More general concept for triangular fuzzy functions is considered, which includes non-regular ones; 3) The concept of triangular fuzzy function is extended for the case of two variables; 4) The existence and uniqueness theorem is established for the fuzzy heat equation, in commonly accepted conditions.

The paper is organized as follows. Following the Introduction, in the Second section, we define triangular fuzzy functions which play a crucial role in our solution method. In the Third section, we describe the fuzzy heat equation and develop the solution method. In the Fourth section, we solve an example to show applicability of the method. Here we also provide a case study. The advantages of the proposed method we discuss in the Fifth Section.

## 2. Triangular fuzzy functions

In this paper, we interpret a fuzzy function as a fuzzy bunch (set) of real functions, where each real function has a certain membership degree. When we refer to value $\tilde{F}(t)$ of such a fuzzy function $\tilde{F}$ at $t$, we mean a fuzzy set (fuzzy number) consisting of values of all the real functions at $t$. If the same value takes place for different functions, the higher membership degree of the functions is assigned to be the membership degree of the value. More formally,

$$\mu_{\tilde{F}(t)}(x) = \alpha \;\Leftrightarrow\; \exists y(\cdot) : \left(\mu_{\tilde{F}}(y) = \alpha \;\wedge\; y(t) = x\right) \;\wedge\; \forall z(\cdot) : \left(\mu_{\tilde{F}}(z) > \alpha \;\rightarrow\; z(t) \neq x\right)$$

where "∧" and "→" are the logical conjunction and implication symbols [22].

Besides, we use the concept of triangular fuzzy functions (a practical case of fuzzy bunch) introduced by Gasilov et al. in Ref. [12].

**Definition 1.** *Let $F_a(\cdot)$, $F_c(\cdot)$, $F_b(\cdot)$ be continuous functions on an interval I.*

$$\mu_{\tilde{F}}(y(\cdot)) = \begin{cases} \alpha, & \text{if } y = F_a + \alpha(F_c - F_a) \text{ and } 0 < \alpha \le 1 \\ \alpha, & \text{if } y = F_b + \alpha(F_c - F_b) \text{ and } 0 < \alpha \le 1 \\ 0, & \text{otherwise} \end{cases}$$

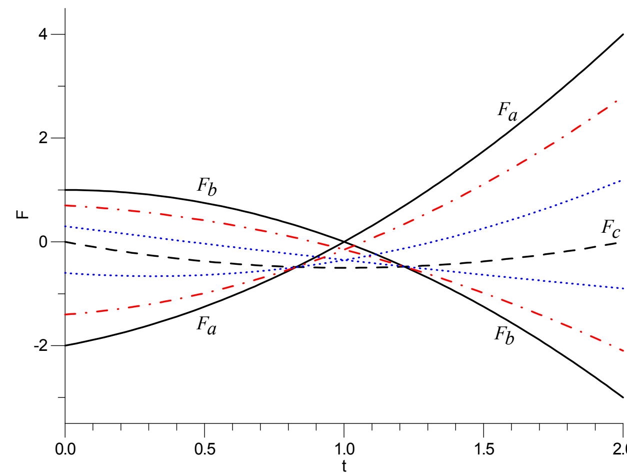


**Fig. 1.** A triangular fuzzy function $\tilde{F} = \langle F_a, F_c, F_b \rangle$ is a bunch of real functions. Functions with membership degrees 0 (the functions $F_a$ and $F_b$), 0.3, 0.7 and 1 (the function $F_c$) are depicted by continuous, dashed-dotted, dotted and dashed lines, respectively.

*The fuzzy set $\tilde{F}$, determined by the membership function*

*is called as triangular fuzzy function and it is denoted by* $\tilde{F} = \langle F_a, F_c, F_b \rangle$.

According to this definition a triangular fuzzy function is a fuzzy set (or, a fuzzy bunch) of real functions. Among them only two functions have the membership degree $\alpha$: the functions $y_1 = F_a + \alpha(F_c - F_a)$ and $y_2 = F_b + \alpha(F_c - F_b)$ (if $F_a$, $F_c$, $F_b$ are pairwise distinct functions).

Geometrically, a bunch $\langle F_a, F_c, F_b \rangle$ consists of two groups of curves linearly distributed between graphics of functions 1) $F_c$ and $F_a$, and 2) $F_c$ and $F_b$.

**Example**. In Fig. 1 we depict the triangular fuzzy function $\tilde{F} = \langle F_a, F_c, F_b \rangle$, where $F_a(t) = t^2 + t - 2$ (membership degree is 0; the curve that is at bottom on $[0,\ 2\sqrt{2}-2]$); $F_c(t) = 0.5t^2 - t$ (membership degree is 1; the dashed curve); $F_b(t) = -t^2 + 1$ (membership degree is 0; the curve that is at upper on $[0,\ (1+\sqrt{7})/3]$).

Functions with membership degrees 0.7 and 0.3 are depicted by dotted and dashed–dotted curves, respectively.

In Fig. 2 we give continuous representation for the triangular fuzzy function under consideration. Here we use a grayscale image, where black color represents the degree of membership of 1 and white represents 0.

We note that, for example, $\tilde{F}(0.5) = (-1.25,\ -0.375,\ 0.75)$ and $\tilde{F}(1.5) = (-1.25,\ -0.375,\ 1.75)$ are "two-sided" triangular fuzzy numbers, while $\tilde{F}(1) = (-0.5,\ -0.5,\ 0)$ is a "one-sided" triangular fuzzy number. (If $a < b < c$ we call the triangular fuzzy number $(a, b, c)$ as "two-sided" triangular fuzzy number, otherwise (if $a = b < c$ or $a < b = c$) as a "one-sided" one.)

For each fixed time $t \in I$, the value of a triangular fuzzy function is a triangular fuzzy number and can be expressed by the following formula:

$$\tilde{F}(t) = \left(\min\{F_a(t),\ F_c(t),\ F_b(t)\}\ ,\ F_c(t),\ \max\{F_a(t),\ F_c(t),\ F_b(t)\}\right)$$

Note that a triangular fuzzy function $\tilde{F} = \langle F_a, F_c, F_b \rangle$ is a fuzzy subset of the universe of continuous functions. Therefore, $\tilde{F}$ is a fuzzy set (bunch) of real functions, where each real function has a certain membership degree. It is not a fuzzy number-valued