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Onset of convection in a vertical porous cylinder with a permeable and conducting side boundary

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A B S T R A C T

The onset of natural convection in a vertical porous cylinder saturated by a fluid is studied. The lateral confinement of the porous cylinder is due to an external porous medium having a permeability much smaller than that of the cylinder. Thus, the vertical side boundary of the cylinder is permeable and constrained by given pressure and temperature distributions. The lower and upper plane boundaries of the cylinder are impermeable walls. The lower wall is subject to a uniform heat flux, while the upper wall has a uniform temperature. The basic motionless state displays a uniform and vertical temperature gradient oriented downward. The linear stability analysis is carried out by using an analytical dispersion relation. The allowed modes of perturbation are determined as solutions of the Helmholtz–Dirichlet problem. First, the natural convection problem is formulated for a circular cylinder. Then, the investigation is generalised to an arbitrary cross-sectional shape of the cylinder. The sample case of an elliptical cylinder is studied in detail, by adopting an analytical solution based on Mathieu functions.

1. Introduction

Natural convection in fluid saturated porous media is a phenomenon with important upshots relative to several applications, ranging from geophysics and planetary science to heat transfer engineering. Hot groundwater flows and lava flows may be induced by natural convection, while this type of convection may come about as a side-effect in the operation of heat exchangers where a porous medium, such as a metal foam, is employed to enhance the heat transfer.

Generally speaking, conditions for the onset of natural convection cells in a porous medium can be determined whenever a vertical temperature gradient oriented downward is established across the medium. This physical setup is usually denoted as *heating-from-below*, and represents the core of the so-called Rayleigh–Bénard convection in a porous medium. Several surveys of the existing literature on this topic are currently available [1–4]. Interesting experimental results on this topic have been obtained by Shattuck et al. [5] and by Howle et al. [6]. The features of the natural convection process arising in a Rayleigh–Bénard system, involving a saturated porous medium, are strongly influenced by the form of the temperature and velocity boundary conditions prescribed at the upper and lower boundaries. Moreover, also the geometry of the vertical sidewall confining the system, and the boundary conditions thereof, are definitely very important.

Linear stability analysis of a Rayleigh–Bénard porous medium is appropriate to determine the threshold, or onset, conditions that may turn a stable, motionless, basic state into a flow state, where natural convection cells set in. Typically, the onset condition is formulated as the Rayleigh, or Darcy-Rayleigh, number being larger than a given critical value. This value is $4\pi^2$ for a saturated porous layer with infinite horizontal width, modelled through Darcy's law, and bounded by isothermal and impermeable walls. A smaller critical value of the Rayleigh number, 27.1, is needed for the onset of convection when the lower bounding wall is kept at a uniform heat flux instead of being isothermal [4]. A lateral confinement by vertical sidewalls yields generally a critical value of the Rayleigh number larger than expected when the porous layer has an infinite horizontal width. In fact, a lateral confinement constrains the possible modes of perturbation, so that neutrally stable modes corresponding to the least possible Rayleigh number are available in an unconfined porous layer, but they may be unavailable when lateral vertical boundaries exist. This feature was recognised by several authors [7–22], modelling the lateral confinement through

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Nomenclature

a	radius of the cylinder
A	compact region in the plane (x,y)
c	half focal distance, Eq. (24)
ce_n,se_n	even and odd Mathieu functions
∂A	boundary of region A
∇_2^2	two-dimensional Laplacian, Eq. (23)
e, o	even, odd modes
$\boldsymbol{e}_z$	unit vector along the z axis
f_n,h_n,f_α,h_α	functions of z, Eqs. (11) and (21)
F	function of x,y, Eq. (19)
$\boldsymbol{g}$	gravity acceleration, $\boldsymbol{g} = -g\,\boldsymbol{e}_z$
H	height of the cylinder
$\Im$, $\Re$	imaginary, real part
J_n	Bessel function of the first kind and order n
k	effective thermal conductivity
K	permeability
m,n	integers
p	difference between pressure and hydrostatic pressure
q_0	wall heat flux
Q_α	eigenfunctions, Eq. (21)
r,θ,z	cylindrical coordinates
R	Rayleigh number, Eq. (4)
s	aspect ratio, Eq. (4)
t	time
T	temperature
T_0	reference temperature
$\boldsymbol{u}$	velocity, $\boldsymbol{u} = (u,v,w)$
x,y	Cartesian coordinates

Greek symbols

α	wave number, Eqs. (11) and (21)
β	thermal expansion coefficient
γ_n,γ_α	exponential growth rate, Eqs. (11) and (21)
ΔT	reference temperature difference
ε	perturbation parameter, Eq. (9)
ζ	parameter, Eq. (17)
η,ξ	elliptical coordinates, Eq. (24)
$\varkappa$	effective thermal diffusivity
μ	dynamic viscosity
λ	parameter, Eq. (17)
ξ_0	boundary value of ξ
ρ	fluid density
σ	volumetric heat capacity ratio
φ_n	function of z, Eq. (14)
χ	elliptic aspect ratio, $\chi = \tanh\xi_0$

Subscripts and superscripts

$\hat{}$	perturbation fields, Eq. (9)
b	basic solution
c	critical value
ext	external porous environment

different geometries: a rectangular box, a circular cylinder, and a pair of coaxial circular cylinders.

In a recent paper [23], the lateral confinement of the porous layer was modelled by permeable plane boundaries where the pressure and temperature distributions are prescribed. That model is further developed in the present paper by dealing with a cylindrical side boundary instead of a pair of plane boundaries. As in the paper by Barletta et al. [23], we will assume the plane horizontal boundaries to be perfectly impermeable. The lower boundary will be considered as subject to a uniform heat flux, while the upper boundary will be considered as isothermal. The combination of isoflux/isothermal boundary conditions in the Rayleigh–Bénard instability of a porous layer was examined by Nield [24]. This choice is alternative to the usual Rayleigh–Bénard setup where both boundaries are regarded as isothermal. Assuming a uniform heat flux instead of a uniform temperature on the lower boundary may be a closer representation of the actual conditions in an experiment where the heating from below is supplied by means of an electric resistance.

Our study will first focus on a circular cylinder and then it will be extended to a cylinder with a general cross-sectional shape. A special attention will be devoted to an elliptical porous cylinder, whose linear stability will be studied analytically by using Mathieu functions to express the normal modes of perturbation.

2. Problem formulation

We aim to analyse the onset of natural convection in a vertical porous cylinder, of height H and with circular cross-section of radius a, saturated by a fluid. Cylindrical coordinates $\boldsymbol{r} = (r,\theta,z)$ are employed, with the z-axis oriented vertically, as shown in Fig. 1. The seepage velocity is denoted by $\boldsymbol{u} = (u,v,w)$, where (u,v,w) are the radial, angular and axial components, respectively.

2.1. The boundary conditions

Let us assume that the lower boundary, $z = 0$, is impermeable and uniformly heated with a heat flux, q_0, while the upper boundary, $z = H$, is impermeable and kept at a uniform constant temperature, T_0. Following the same assumptions adopted by Barletta et al. [23], the side boundary at $r = a$ is considered as perfectly permeable and in perfect thermal contact with an external porous reservoir. The reservoir, saturated by the same fluid contained in the cylinder, is in a steady state with a vertical thermal stratification. All details about the modelling of the side boundary can be found in Barletta et al. [23]. An essential feature of the external porous reservoir is that its permeability must be much smaller than that of the porous medium contained in the cylinder. This assumption serves to ensure that, when the Rayleigh number becomes sufficiently large for the onset of the instability in the porous cylinder, its value is still subcritical in the external porous reservoir.

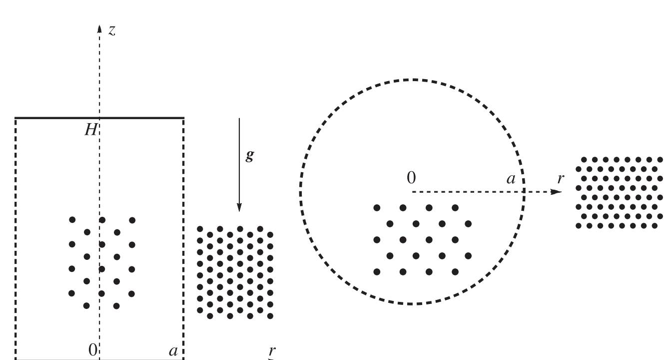

Fig. 1. A sketch of the porous cylinder.