International Journal of Thermal Sciences 96 (2015) 268–276

Contents lists available at ScienceDirect

International Journal of Thermal Sciences

journal homepage: www.elsevier.com/locate/ijts

An iterative algorithm for the stable solution of inverse heat conduction problems in multiply-connected domains

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A R T I C L E I N F O

Article history:
Received 13 February 2014
Received in revised form
16 January 2015
Accepted 23 February 2015
Available online 29 March 2015

Keywords:
Cauchy problem
Inverse problem

A B S T R A C T

The Cauchy problem for inverse heat conduction in a multiply-connected domain Ω was solved in this paper. To obtain this, an iterative algorithm which minimized a functional comprising the homogeneity of the outer domain surface temperature and the temperature gradient within the domain Ω was developed. Calculations were made for the known distribution of the heat transfer coefficient and surrounding temperature on the outer boundary of the domain, disturbed by the random error $\delta[\%] = 0, 1, 5, 10$. The influence of temperature gradient on time and accuracy of calculations was investigated. Taking into consideration the temperature gradient in the functional being minimized in the analytical process shortens the time of calculations and reduces oscillations of temperature and heat flux distributions on the inner boundary of the multiply-connected domain. An example of application of the algorithm presented in this paper can be the optimization problem in cooling of gas turbine blades.

1. Introduction

Inverse problems belong to the class of ill-posed problems in the Hadamard's sense [5]. There are many algorithms for solving such problems; the most important among them are: the Tikhonov regularization [8], future-sequential regularization method [2], SVD algorithm (singular value decomposition) [7] and iterative algorithms [1,10].

The inverse problem being solved in this paper is significant from a technical point of view since it concerns the problem of cooling gas turbine blades. A turbine blade profile with cooling channels distributed inside is an example of a multiply-connected region of a complex structure. For design problems, the temperature distribution and the heat transfer coefficient α on the outer boundary of the blade are often known from external flow and heat transfer analysis and the temperature and heat flux distributions on the walls of cooling channels (the Cauchy problem) should be determined to assess appropriate cooling technologies.

To address this problem ref. [4] focused on searching for temperature and heat flux distributions in cooling channels of the blade for a constant temperature on the blade's outer boundary and given heat fluxes. For the functional, whose minimum was determined in the optimization process, the following form was applied:

$$J = \frac{1}{2} \int\limits_{\Gamma_{out}} (T - T_o)^2 ds \tag{1}$$

where $\boldsymbol{T_o}$ is the temperature given on the outer boundary, which should be achieved.

The results presented in Ref. [4] concerned the C3X blade with ten cooling channels of circular cross-section [6]. Temperature distribution on the blade's outer boundary oscillated strongly in those parts of the boundary which were in the neighbourhood of the blade's cooling channels. It caused strong temperature and heat flux oscillations on the walls of the blade's cooling channels. Inverse solutions obtained in such a way had to be averaged for each channel to obtain meaningful interpretations.

This indicates that there is a need to modify the functional (1) and to examine the influence of random errors for the boundary conditions on the stability of the solution to the inverse problem with a modified functional. Some investigations based on the functional of the form (1) were presented in Ref. [3].

Oscillations effects of the solution to the inverse problem from the investigations in Ref. [4] may be dampened by applying the

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http://dx.doi.org/10.1016/j.ijthermalsci.2015.02.016
1290-0729/

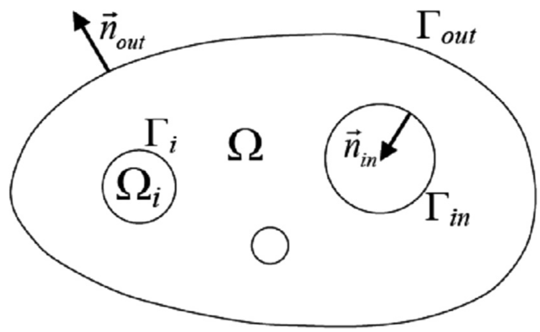

Fig. 1. Multiply-connected domain Ω bounded by the boundary $\Gamma = \Gamma_{out} \cup \Gamma_{in}$.

following functional, defined for a multiply-connected domain, Fig. 1:

$$J = \frac{1}{2} \int\limits_{\Gamma_{out}} (T - T_o)^2 ds + \frac{\gamma}{2} \int\limits_{\Omega} (\nabla T)^2 d\omega, \quad \gamma > 0 \tag{2}$$

where T_o is the temperature given on the outer boundary of the region and γ is a regularization parameter.

Temperature oscillations influence the value of the functional (2). Minimizing the functional (2) results in smoothing the function of temperature $\boldsymbol{T}$ in the iteration process. Here it is assumed that on the boundary Γ_{out} not only the temperature (the first-type boundary condition) is given, but also the third-type boundary condition with the known heat transfer coefficient α leading to the Cauchy problem for the heat conduction process within the turbine blade.

The Cauchy problem for the Laplace's equation was solved, among others, in papers [11–14], and the Cauchy problem for the Laplace's equation in the multiply-connected domain – in papers [15–23].

2. Iterative algorithm

The Cauchy problem for a multiply-connected domain (Fig. 1) with constant thermal conductivity is related to solving the Laplace equation in the domain Ω

$$\Delta T = 0 \tag{3}$$

with the boundary conditions of the first and third type on the boundary Γ_{out}:

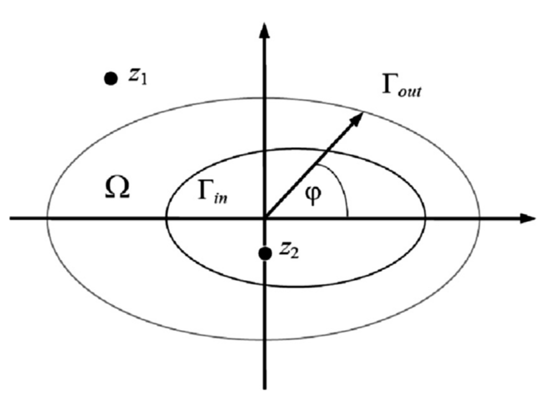

Fig. 2. Domain Ω – Elliptical ring with shifted inner boundary.

$$\begin{aligned} &T|_{\Gamma_{out}} = T_o \\ &-\lambda \frac{\partial T}{\partial n}\Big|_{\Gamma_{out}} = \alpha\left(T|_{\Gamma_{out}} - T_f\right) \end{aligned} \tag{4}$$

where T_o –outer boundary temperature, λ – thermal conductivity of the blade material, α – heat transfer coefficient, and $\boldsymbol{T_f}$ – the temperature of the surrounding fluid at the outer boundary of the domain Ω,.

Solving such ill-posed inverse problems may be performed by solving in replacement consecutive direct problems:

$$\begin{aligned} &\Delta T = 0 \\ &\frac{\partial T}{\partial n}\Big|_{\Gamma_{out}} = -\frac{\alpha}{\lambda}\left(T|_{\Gamma_{out}} - T_f\right) \\ &\frac{\partial T}{\partial n}\Big|_{\Gamma_{in}} = g \end{aligned} \tag{5}$$

Function **g**, determined on the boundary Γ_{in} of the domain Ω, changes iteratively, so that the functional (2) will obtain its minimum within the limits of the iterative process. For this purpose a variation of this functional needs to be considered.

The variation of the functional (2) is equal to:

$$\delta J[g] = \int\limits_{\Gamma_{out}} (T(g) - T_o)\delta T(g) ds + \gamma \int\limits_{\Omega} \nabla T(g) \nabla \delta T(g) d\omega \tag{6}$$

Using the relation:

$$\int\limits_{\Omega} \nabla T \nabla \delta T d\omega = \int\limits_{\Gamma_{out}} \frac{\partial T}{\partial n} \delta T ds + \int\limits_{\Gamma_{in}} \frac{\partial T}{\partial n} \delta T ds$$

and taking into consideration the boundary conditions (5) for the function $\boldsymbol{T}$ and the boundary conditions for the function $\delta \boldsymbol{T}$

$$\begin{aligned} &\frac{\partial \delta T}{\partial n}\Big|_{\Gamma_{out}} = -\frac{\alpha}{\lambda} \delta T|_{\Gamma_{out}} \\ &\frac{\partial \delta T}{\partial n}\Big|_{\Gamma_{in}} = \delta g \end{aligned} \tag{7}$$

we obtain

$$\int\limits_{\Omega} \nabla T(g) \nabla \delta T(g) d\omega = -\int\limits_{\Gamma_{out}} \frac{\alpha}{\lambda} T(g) \delta T(g) ds + \int\limits_{\Gamma_{in}} T(g) \delta g ds$$

Finally, the variation of the functional (2) takes the following form:

$$\delta J[g] = \int\limits_{\Gamma_{out}} \left[\left(1 - \frac{\gamma\alpha}{\lambda}\right) T(g) - T_o\right] \delta T(g) ds + \gamma \int\limits_{\Gamma_{in}} T(g) \delta g ds \tag{8}$$

To determine the variation of the function δT on the boundary Γ_{out} as occurred in Equation (8), we need an auxiliary function p (adjoint with δT), satisfying the Laplace equation. For the functions δT and p the following identity is true:

$$\begin{aligned} &\int\limits_{\Gamma_{out}} \frac{\partial \delta T}{\partial n} p ds + \int\limits_{\Gamma_{in}} \frac{\partial \delta T}{\partial n} p ds = \int\limits_{\Gamma_{out}} \delta T \frac{\partial p}{\partial n} ds + \int\limits_{\Gamma_{in}} \delta T \frac{\partial p}{\partial n} ds \\ &\int\limits_{\Gamma_{out}} \left(\frac{\partial p}{\partial n} + \frac{\alpha}{\lambda} p\right) \delta T ds = \int\limits_{\Gamma_{in}} \delta g p ds - \int\limits_{\Gamma_{in}} \delta T \frac{\partial p}{\partial n} ds \end{aligned} \tag{9}$$