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Solving nonlinear heat transfer problems using variation of parameters

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A B S T R A C T

Nonlinear problems arise in many heat transfer applications, and several analytical and numerical methods for solving these problems are described in the literature. Here, the method of variation of parameters is shown to be a relatively simple method for obtaining solutions to four specific heat transfer problems: 1. a radiating annular fin, 2. conduction-radiation in a plane-parallel medium, 3. convective and radiative exchange between the surface of a continuously moving strip and its surroundings, and 4. convection from a fin with temperature-dependent thermal conductivity and variable cross-sectional area. The results for each of these examples are compared to those obtained using other analytical and numerical methods. The accuracy of the method is limited only by the accuracy with which the numerical integration is performed. The method of variation of parameters is less complex and relatively easy to implement compared to other analytical methods and some numerical methods. It is slightly more computationally expensive than traditional numerical approaches. The method presented may be used to verify numerical solutions to nonlinear heat transfer problems.

1. Introduction

Modeling of heat transfer processes often results in the development of nonlinear differential equations. Specifically, application of an energy balance to a system frequently results in a nonlinear differential equation that governs the temperature field in the system. Phenomena that give rise to nonlinear differential equations include radiative exchange between surfaces, temperature-dependent properties, modeling the dependence of a convective heat transfer coefficient on temperature, and the coupling of the energy equation with the total radiative heat flux in radiatively participating media. In general, nonlinear differential equations do not have analytical, closed-form solutions, so they are typically solved using numerical methods. Best practice requires verification of numerical solutions, and the analytical approach described in this paper is a tool for verifying algorithms used to obtain numerical solutions of nonlinear heat transfer problems.

Other analytical methods have been used to solve the nonlinear differential equations that arise in heat transfer applications. One of the most common nonlinear heat transfer problems comes from the analysis of extended surfaces. Abbasbandy and Shivanian [1] obtained an exact analytical solution to the convective fin problem in which the local convection coefficient along the fin surface has a power-law-type dependence on the local temperature difference between the fin and the surrounding fluid. Arslanturk [2] used the Adomian decomposition method (ADM) to analyze a convective fin with temperature-dependent thermal conductivity while Moradi and Ahmadikia [3] solved a similar problem for fins of variable cross-sectional area using the differential transformation method (DTM). Aziz and Khani [4] used the homotopy analysis method (HAM) to solve the nonlinear equation describing the temperature distribution in a continuously moving radiative-convective fin with temperature-dependent thermal conductivity. The temperature distribution in convective annular fins with temperature-dependent thermal conductivity has been found by Ganji et al. [5] using the homotopy perturbation method (HPM) while that of radiating annular fins has been found using a Green's function approach [6–8]. The advantages of many of these analytical approaches over numerical methods are their direct applicability to both linear and nonlinear equations without requiring linearization, discretization, or perturbation [3]. However, implementation of these complex methods requires the use of infinite power series.

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In addition to extended surfaces, nonlinear differential equations arise when analyzing the energy equation in a radiatively participating medium. These problems are highly nonlinear because the energy equation requires the total radiative heat flux, which is found by solving the integro-differential radiative transfer equation (RTE) [9]. These problems are encountered in the analysis of combustion chambers, rocket nozzles, high-temperature heat exchangers, translucent glass or ceramic coatings, porous insulation, heat treatment of glass plates, and the drawing of optical fibers. The combined conduction-radiation problem has been solved using an integral transformation method [10,11], the finite element method [12,13], the finite difference method [14,15], the finite volume method [16,17], the lattice Boltzmann method [18–20], and finite strip theory [21].

The variation of parameters method has been primarily used to solve linear, nonhomogeneous differential equations, but application of this method to solve nonlinear differential equations has been described previously [22–25]. These publications focus on general mathematical aspects of the solution, particularly application of the method to problems involving an inhomogeneity that is a function of the dependent variable. The contribution of this work is a demonstration that the method of variation of parameters is a relatively simple method of obtaining exact solutions to the nonlinear differential equations that arise in a variety of heat transfer applications. In addition to its intrinsic value as an analytical solution procedure, this approach may be used to verify solutions obtained using more computationally efficient numerical methods.

Following a brief overview of how the method of variation of parameters may be used to obtain exact solutions of nonlinear equations, application of the method is illustrated by solving models derived from analysis of four heat transfer applications. Application of the method to the models developed for the first three problems results in an integral equation that is solved using numerical quadrature. Although numerical methods are used, it should be noted that numerical integration may be performed to an arbitrary degree of precision, so while not closed-form, these solutions can be considered exact. The last example requires finite difference approximations of derivatives, so it is not an exact solution. These solutions are compared with solutions obtained using other analytical or numerical methods. These examples demonstrate that variation of parameters is an easily implemented method of solving the nonlinear differential equations that result from the analysis of a wide range of heat transfer applications.

2. Method of variation of parameters

Consider the following second order partial differential equation

$$\frac{d^2y}{dx^2} + C\frac{dy}{dx} + Dy = f(x,y), \quad x_1 \le x \le x_2 \tag{1}$$

where C and D are constants and f is a nonlinear function of the independent variable x and the dependent variable y. The solution to the boundary value problem in Equation (1) consists of the sum of the complementary and particular solutions. The complementary solution is the solution to the homogeneous equation corresponding to Equation (1) that may be found using traditional solution techniques [26]. The complementary solution is

$$y_c(x) = c_1y_1(x) + c_2y_2(x) \tag{2}$$

where c_1 and c_2 are constants and y_1 and y_2 form a fundamental set of solutions of the homogeneous equation. The particular solution [26] is

$$y_p = u_1(x,y)y_1(x) + u_2(x,y)y_2(x) \tag{3}$$

where

$$u_1(x,y) = -\int_{x_1}^{x} \frac{f(t,y)y_2(t)}{W(y_1,y_2)}dt \tag{4}$$

$$u_2(x,y) = \int_{x_1}^{x} \frac{f(t,y)y_1(t)}{W(y_1,y_2)}dt \tag{5}$$

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \tag{6}$$

The solution to Equation (1) is therefore

$$y(x) = c_1y_1(x) + c_2y_2(x) + u_1(x,y)y_1(x) + u_2(x,y)y_2(x) \tag{7}$$

The constants c_1 and c_2 are found from the boundary conditions. Because u_1 and u_2 are functions of the dependent variable, y, an iterative approach using the method of successive approximations is required to determine y, and numerical integration of Equations (4) and (5) is required. Note that f may be a function of the derivatives of y. In such cases, the derivatives of y are approximated using finite difference equations.

In order to assess the accuracy and computational efficiency of this approach relative to the accuracy and computational efficiency of numerical methods, the variation of parameters solution and the finite difference solution of Equation (8) are compared to its exact solution [27].

$$y'' = x + y \tag{8}$$

Note that in this equation, the inhomogeneity is a function of both the dependent and independent variables. If the y on the right side is moved to the left side, this equation becomes a simple ordinary, linear, nonhomogeneous differential equations which may be solved exactly. Solutions were obtained for various boundary conditions. The average error in the variation of parameters solution ranged from 2.6% when 10 steps were used in the numerical quadrature to 0.0007% when 1000 steps were used in the numerical quadrature. Clearly, the accuracy of the variation of parameters approach is limited only by the accuracy of the numerical integration. Complete details regarding the comparisons between the exact solution and the variation of parameters solution are available in Ref. [27].

This problem was also solved using a finite difference method. For a given step size, the average error between the exact solution and that found using finite difference methods was the same as the error between the exact solution and the variation of parameters solution to four significant figures. The complexity of the implementation of the method of variation of parameters was comparable to that of the finite difference approach and the computational time required by the method of variation of parameters method was slightly larger than that required by the finite difference approach.

Example 1. Radiating Annular Fin

Heat flux measurements are important in many processes and procedures. These include monitoring ovens, furnaces, and kilns, flammability testing, measuring the thermal properties of