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Thermal quadrupole approaches applied to improve heat transfer computations in multilayered materials with internal heat sources

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ABSTRACT

An extension to the classical quadrupole method is proposed which allows computing temperature and heat fluxes anywhere inside a multilayer material containing localized and/or distributed heat sources. The distributed heat sources are not limited to being uniform in each layer. By using the superposition principle and through a treatment which depends on the relative position of the heat sources and the observation point, we get a closed-form analytical expression for the temperature/flux vector which yields stable results over an arbitrary time scale. This source-sampled quadrupole method is based on the *transfer* formulation related to a T-scheme two-port network representation. We propose another approach based on the *impedance* formulation related to the same T-scheme two-port network representation; it leads to a global *impedance* matrix formulation which provides first the heat flux vector and then the temperature vector. Alternatively we also propose an approach based on the *admittance* formulation related to a Π-scheme two-port network representation with admittances; it leads to a global *admittance* matrix formulation which provides first the temperature vector and then the heat flux vector. Both *impedance* and *admittance* matrix formulations are easier to program than the source-sampled quadrupole method but their computing time is slightly higher. The three proposed methods are illustrated on two four-layer slabs, one with a uniform heat source distribution in each layer and the other one with exponential heat source profiles.

1. Introduction

For the computation of temperature and/or heat flux in multilayered time-varying linear systems, the quadrupole method (also known as the two-port network method) is a well appropriate analytical method. It is based on the fact that the temperature-flux vectors on both sides of an homogeneous opaque slab are related through a matrix product involving a 2×2 quadrupole matrix [1]. This matrix product involves temperature and heat fluxes in a transformed space, either the Fourier space if one is interested in the time-periodic regime, or the Laplace space if one is interested in the transient regime. Multiplying together the quadrupole matrices corresponding to each layer of a multilayer system yields the global quadrupole matrix of the stack which relates temperature-flux vectors on both sides of it. Then, by taking into account the boundary conditions and finally by performing a Fourier inversion, resp. a Laplace inversion, one gets the requested external temperatures or fluxes. Internal temperatures and fluxes can be computed accordingly by multiplying the temperature-flux vector obtained previously at one external surface by the quadrupole corresponding to the subsystem extending from this surface to the requested depth. This is the "natural" approach, which however presents some drawbacks, as will be seen later.

Fundamentals of the quadrupole method together with numerous applications are described in Ref. [2]. Since the first presentation of the quadrupole method for one-dimensional transfer in Ref. [1] a large number of applications were published. They include thermal characterization [3–7] non destructive testing [8–10] heat transfer analysis in buildings [11–13], and in electronic systems [14], for most recent.

The quadrupole method can also be used for multidimensional heat transfer analysis provided one applies an additional integral transform which depends on the geometry, the eventual symmetries and the boundary conditions: two particular cases are the Fourier transform for Cartesian geometry and the Hankel transform for cylindrical geometry [14–21].

For multidimensional heat transfer in heterogeneous media with one-dimensional variation of thermal properties, a semi-numerical

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Nomenclature

a	thermal diffusivity $m^2\ s^{-1}$
A_i, B_i, C_i, D_i	quadrupole coefficients (Eq. (2))
b	thermal effusivity $W\ s^{1/2}\ m^{-2}\ K^{-1}$
ρ	density $kg\ m^{-3}$
C	heat capacity $W\ s\ kg^{-1}\ K^{-1}$
g	heat source power density $W\ m^{-3}$
G	Laplace transform of heat source power density $W\ s\ m^{-3}$
h	heat transfer coefficient $W\ m^{-2}\ K^{-1}$
f	frequency s^{-1}
l	thickness m
$\mathbf{M}$	quadrupole matrix
n	number of (finite) layers
p	Laplace variable s^{-1}
R	Thermal resistance $W^{-1}\ m^2\ K$
$\mathbf{S}$	source vector of the quadrupole
t	time s
T, T_0	temperature; initial temperature K
Y	admittance
$\mathbf{Y}$	admittance matrix for a layer
$\overline{\mathbf{Y}}$	admittance matrix for a multilayer
Y_1, Y_2, Y_3	thermal admittances of Π-network
z	space location (1D) m
Z	impedance
$\mathbf{Z}$	impedance matrix for a layer
$\overline{\mathbf{Z}}$	impedance matrix for a multilayer
Z_1, Z_2, Z_3	thermal impedances of T-network
β	absorption coefficient m^{-1}
Λ	common denominator (Eq. (14))
φ	heat flux density $W\ m^{-2}$
ϕ	transform of heat flux density $W\ s\ m^{-2}$
λ	thermal conductivity $W\ m^{-1}\ K^{-1}$
μ	thermal penetration depth m
θ	Laplace transform of temperature K s
Θ	tension source of the quadrupole K s
Φ	current source of the quadrupole $W\ s\ m^{-2}$
ω	pulsation rad s^{-1}

Superscript

b	relative to backward transfer formulation
f	relative to forward transfer formulation
Y	relative to admittance quadrupole formulation
Z	relative to impedance quadrupole formulation
$\tilde{A}_i$	after division of A_i by $\exp(l_i/\mu_i)$
*	localized (heat source)
+	right from a localized heat source (for heat flux density)
−	left from of a localized heat source (for heat flux density)

Subscript

F	front face (left on Fig. 1)
i	layer or interface number (source position)
j	layer or interface number (observation position)
k, m	after quadrupole multiplication from layer k to layer m
in	input
out	output
R	rear face (right on Fig. 1)
r	reflective
t	transmissive

general solution was also proposed, based on a semi-gridding approach [22,23].

An extension to the multiblock multilayer case using matrices of spectrum conversion was proposed in Refs. [24,25]. This method is suited to solving heat transfer through power electronic components [24,25] and in media containing a non uniform thermal resistance (non-destructive testing application) [26].

The quadrupole method was also adapted for analysing convection–diffusion problems [27,28] and coupled heat and mass transfer problems [29]. Water and heat diffusion in soils was also considered in Ref. [30] where special quadrupoles where developed for taking into account continuous profiles for thermal and hydraulic properties. The functional used for describing these profiles led to quadrupoles of the same type as those developed in Ref. [31] for modelling the advective and diffusive transport of a passive solute in porous media. Quadrupoles for layers showing a linear effusivity profile were described in Ref. [32]. Moisture transport in soil and plant was also simulated with tailored quadrupoles in Refs. [33,34].

An upgrade was proposed for addressing the case of materials containing heat sources [35]; the thermal quadrupole scheme is then modified through the addition of a tension source and a current source [36,2]. The thermal relaxation of an initial non uniform temperature distribution can also be modelled by considering a distribution of specific internal heat sources. Thermal quadrupoles with internal heat sources were considered for modelling a laser photothermal radiometry experiment on a semitransparent sample [37–39], heat transfer within monolithic solid-state microcoolers [40], heat dissipation in the soft starter of an induction motor [14], microchannel reactors [27], moisture and heat diffusion in soils [30,34].

Although being elegant, the quadrupole method faces some drawbacks: numerical instabilities and overflows may occur when computation is attempted for a time value being small with respect to the higher diffusion time of the layers, i.e. $t << \max(l_i^2/a_i)$, where l_i is thickness and a_i is diffusivity of layer i. These peculiarities are due to the presence of hyperbolic functions with too high arguments. A well known remedy consists in factorizing each quadrupole matrix by the corresponding exponential function of positive argument [41]: for a single layer with heat sources at both surfaces and with a uniform internal source, all these exponential vanish in the temperature expression at front and rear face of the layer (only remain exponential functions of negative argument at the numerator). Temperature can then be calculated with high precision over an arbitrary time scale. For a multilayer with internal heat sources, a global matrix approach was proposed in Ref. [41] which also leads to a solution deprived from exponential functions with positive arguments. It is however restricted to *uniform* heat sources in each layer. It also implies an intermediate step with the computation of a series of fictitious temperatures before getting the vector of interface temperatures.

We will address the problem of modelling the transient thermal evolution of a 1D multilayer submitted to internal and external heat sources, independent of temperature. We will present three alternative methods sharing the following properties:

- they are robust: temperature and flux can be calculated at any depth over an arbitrary time scale;
- they don't require the computation of fictitious temperatures;
- the distributed heat sources are not restricted to being constant in each layer.