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Scaling relations of branching pulsatile flows

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ABSTRACT

Apparently complex flow structures obey to scaling relations that enable to make it viable the study of their configuration and flow dynamics. This is the case of flow structures that exhibit several branching levels and are thought to perform optimally.

Here we present scaling relations of diameters and lengths of branching cylindrical channels with pulsatile flows, and compare them with other relations published in the literature. It is shown that, under constant global volume of the flow tree, and for zero pulse frequency these scaling relations reduce to Murrays's law of consecutive diameters. Optimal scaling depends on pulse frequency, distensibility of the channel walls, and asymmetry of the daughter vessels. In case that in addition to global volume of the flow tree, the pressure head is also kept constant, a similar scaling law of channel lengths emerges that holds together with the law of diameter scaling. The effect of channel distensibility is shown to be somehow important, such that for achieving optimal performance (lowest impedance) channels with lower relative distensibility must have their diameter increased. Results are compared with those of other models for the case of some arteries.

1. Introduction

Murray's Law [1–3] which states that the "cube of the radius of a parent vessel equals the sum of the cubes of the radii of the daughters" stays as a landmark scaling law of geometries of branching channels with non-turbulent flows (see Fig. 1). It was originally proposed by Cecil D. Murray (1926) for the circulatory and respiratory systems, yet later on has been proved to hold for every branching laminar flow [3,4].

Murray stated in his original work [2] that physiologic organization should be based on principle and pointed out minimum work and balanced cooperation of the organs in the body as the best candidate for such a principle. Sherman [3] provided a full derivation of Murray's law based on that principle. Allometric scaling laws are common in biology and, with the purpose of their explanation, approaches have been developed based on optimal performance of the whole system, either trough minimization of energy dissipation [5] or through flow configuration that enables maximum flow access [6]. West and co-workers [5] presented a general model of allometric scaling relations (WBE model) in that the ratio between the diameters of consecutive arteries, D_{k+1}/D_k, is $n^{-1/2}$ for arteries, and $n^{-1/3}$ for small vessels (n stands for branching ratio), regardless of the length of the vessels.

Murray's Law has also been considered in the context of engineered systems. About a decade ago, Bejan and coworkers [4] proved that Murray's law may be deduced from a general principle – the Constructal Law (1997) – which states: "For a finite-size system to persist in time (to live), it must evolve in such a way that it provides easier access to the imposed currents that flow through it." (see for instance Ref. [7]). Said another way, Constructal Law entails evolution of flow architecture in such a way that under the existing constraints the distribution of flow resistances evolves in time to achieve minimum global flow resistance.

Under the conjecture that Nature has optimized in time the living structures, Reis and coworkers [8] applied both Murray's Law and Constructal Law to successfully anticipate some architectural features of the lung tree. More complete information about the successful application of the Constructal Law may be found in Bejan [6], Reis [9], and Bejan and Lorente [10].

However, we note that with respect to optimal performance Mauroy et al. [11] have put forward the idea that "the optimal system is dangerously sensitive to fluctuations or physiological

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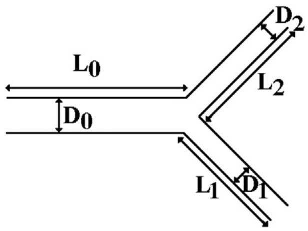

Fig. 1. Branching channels with distensible walls (D – diameter; L – length).

variability, such that physical optimality cannot be the only criterion for design".

With respect to optimal scaling in asymmetric branching, Bejan [7] has shown that

$$\frac{D_1}{D_0} = \left(1+\xi^3\right)^{-1/3}, \frac{D_1}{D_2} = \frac{L_1}{L_2} \tag{1}$$

where $D_2/D_1 = L_2/L_1 = \xi$, is the asymmetry factor of daughter vessels, and the subscripts 0, 1, 2 represent the parent and each one of the daughter vessels, respectively. The Eq. (1) which relates homothety coefficient with asymmetry factor further adds to the study of scaling in asymmetric flows, which are shown to be important for achieving optimal performance of flow trees [12].

In the following we will further extend this analysis to find out the scaling relations of branching pulsatile flows.

2. Pulsatile flows

Flows that develop in circulatory trees are ubiquitous in Nature. In some animals, namely the vertebrates, blood is rhythmically pumped through the entire body at a broad range of pulse rates. It is recognized that pulsatile flow performs best than continuous flow because it induces lower overall resistances [13] and also better blood perfusion [14].

The most complete model of pulsatile flows, was put forward by Womersley [15] who solved Navier–Stokes equation in channel with elastic walls and periodic pressure forcing, and provided formulas for the pressure wave, and the radial and longitudinal components of the velocity field in the arteries. This work that stays as a landmark in the field was used as one of the basis of the WBE model [5].

Since then, other works have appeared that modeled pulsating flows in rigid channels [16]. Noteworthy are those of Nield and Kuznetzov [17], Siegel and Perlmutter [18] and Faghri et al. [19], albeit these studies were also carried out under the "rigid channel" assumption. Models using analogy with electric circuits date back to about several years ago. Remarkable by its complexity are those of Tsitlik et al. [20], Avolio [21], or recently that of Mirzaee et al. [22].

With the purpose of optimizing branching structures with pulsatile flows, in what follows we will further explore the parallel RC model. Though Womersley's equations describe pulsatile flows accurately, they are quite complex, and not easy to handle analytically in the study of branching vessels. This is why we use an RC model as a suitable description of pulsatile flow. In this model the flow induced by the pressure wave "charges the capacitor" (the arterial elastic walls) while it is braked by a "Poiseuille resistance" in the flow direction. The rationale for using Poiseuille flow, rather than considering a more complex model based on the Navier–-Stokes equation is explained in the following.

Let us start from Navier–Stokes equation for unidirectional flow: $\partial u/\partial t + u.\mathrm{grad}\, u = -\rho^{-1}\mathrm{grad}\, P + \nu\, \mathrm{lap}\, u$. In the case of pulsatile flow in arteries, the inertial terms may be discarded because they are, at least, of one order of magnitude smaller than the other terms, as it is shown through scale analysis. In this way, let u denote average blood velocity, τ characteristic time related to pulse wave frequency, L_c the characteristic length in the flow direction, D vessel diameter, ρ blood density, ΔP pressure variation along the vessel and ν blood kinematic viscosity. Then, by assuming the following scale values for large arteries: u~10^{-1} ms^{-1}, τ~1 s, L_c~1 m, D~10^{-3} m, ΔP~10^3 Pa and ν~10^{-5} m^2 s^{-1}, the orders of magnitude (in ms^{-2}) of the terms in the Navier–Stokes equation are: $\partial u/\partial t$~10^{-1}, $u.\mathrm{grad}u$~10^{-2}, $\rho^{-1}\mathrm{grad}P$~1, $\rho^{-1}\mathrm{lap}\, u$~1, therefore justifying the use of Poiseuille flow as a first approach in the study of the human arterial system. Models that include the term $\partial u/\partial t$ lead to greater complexity in the calculations but did not cause a change in the conclusions. For example, the RLC model developed by Jager and co-workers [23] accounts for the "sleeve effect", which arises from the interaction between viscous and inertial terms in the Navier–Stokes equation. However, in the same study [23] it was shown that the "sleeve effect" is important in some arteries at frequencies higher than 15 rad s^{-1}, which is somehow beyond the normal range of the human pulse frequency.

In real systems, pressure waves of some frequency travel all along the circulatory trees. Energy travels in the form of enthalpy plus mechanical energy of the bulk fluid, and in the form of elastic energy of the vessel walls.

As the basis for building up a model of a pulsatile flow driven by a pressure difference ΔP in a vessel of length L and diameter D, one starts from the Hagen–Poiseuille equation in the form:

$$I = k_A^{-1} L^{-1} D^4 \Delta P, \tag{2}$$

where I is current (m^3 s^{-1}), $k_A = 128\mu\pi^{-1}$, in which μ is dynamic viscosity of the fluid. In pulsatile flow, both ΔP and D are functions of time, and therefore the same happens with the conductance $K_p = k_A^{-1}L^{-1}D^4$. In what follows the variables D,L,V standing for geometric features of vessels with pulsatile flow represent values averaged over a cycle. In this way, as a first approximation we will consider the actual conductance in the channel as the sum of the average conductance (corresponding to diameter D) plus the deviation corresponding to diameter variation with pressure, i.e.

$$K_p(t) = K + \overline{K}' = k_A^{-1} L^{-1} D^4 \left(1 + 2\beta (\mathrm{d}P/\mathrm{d}t)_0 \Delta t\right), \tag{3}$$

where $\beta = (2/D)(\mathrm{d}D/\mathrm{d}P)$ is the distensiblity coefficient and Δt is the time elapsed after the channel diameter has reached the average value. The Eq. (3) shows that the conductance is the sum of two terms: the first one corresponds to the inverse of the usual resistance while the second one is equivalent to the inverse of a capacitive resistance. This aspect is made clearer if we consider $I(t) = K_p(t)\Delta P(t)$ together with Eq. (3) to obtain:

$$I \approx k_A^{-1} L^{-1} D^4 \Delta P + 2k_A^{-1} L^{-1} D^4 (\Delta t) \beta \Delta P (\mathrm{d}P/\mathrm{d}t)_0, \tag{4}$$

Eq. (4) shows that the flow in a channel with elastic walls is composed of two terms: one corresponds to a resistive current,

$$I_r = k_A^{-1} L^{-1} D^4 \Delta P, \tag{5}$$

while the other matches up a capacitive current,

$$I_c = 2I_r (\Delta t) \beta (\mathrm{d}P/\mathrm{d}t)_0. \tag{6}$$

with capacitance $C = 2I_r(\Delta t)\beta$, (see Fig. 2).