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A novel analytical method for heat conduction in convectively cooled eccentric cylindrical annuli

Ayhan Yılmazer*, Cemil Kocar

Department of Nuclear Engineering, Hacettepe University, Beytepe, 06800 Ankara, Turkey

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ABSTRACT

Heat conduction equation through a heat generated eccentric cylindrical annulus with the inner surface kept at a constant temperature and the outer surface subjected to convection is analytically solved in bipolar coordinates using the Green's function method. Since it is not possible to find an analytical Green's function to the conduction equation in bipolar coordinates for an eccentric annulus subject to boundary condition(s) of third type (convection), a novel method treating the same problem as a second type boundary value problem is devised. The method has first been applied to heat generating eccentric annuli and results have been compared to the results of computational fluids dynamics (CFD) code FLUENT. Perfect agreement was observed for various geometrical configurations and a wide range of Biot number. Then, heat transfer through eccentric annuli without heat generation was considered. Variation of heat dissipation with radii ratio was studied and a very good agreement with the literature has been observed. A simple approximate analytical expression for the heat transfer rate is derived using first term (zero-order) approximation. It has been demonstrated that this expression gives very accurate results for a wide range of geometrical configurations and Biot number.

1. Introduction

The solution of the problems related to the annular regions between cylinders has been long considered in classical electrodynamics, heat transfer, fluid mechanics, and reactor physics, etc. Among these, analysis of conduction heat transfer for concentric annular cylinders is well developed. Since three types of boundary conditions could be imposed on each surface of the annulus there are nine different conduction problems corresponding to the possible combinations of boundary conditions on two boundaries. Boundaries could be isothermal (first type boundary condition), could be subjected to prescribed heat fluxes (second type boundary condition) or could be subjected convection by a fluid at an ambient temperature (third type boundary condition). Exact solutions of those linear conduction problems for uniform boundary conditions are presented by Özisik [1] for any functional form of source distribution.

Eccentric cylindrical annuli in which boundaries are relatively displaced radially are encountered in many heat transfer applications. The eccentricity could be due to the design or could arise from manufacturing margins and operational conditions. Regardless the reason, the eccentricity may have a considerable effect on the performance of the system. For example, Kundu and Das [2] have studied the performance of the eccentric annular fins for a variation of eccentricity, radii ratio, and Biot number as well as the maximum base temperature. They showed that for a variable base temperature, eccentric annular disc fins had a superior performance compared to the concentric fins of equal volume. As discussed by El-Shaarawi and Mokheimer [3], during operation eccentricity might occur in underground electric cable systems, cylindrical solar collector system, and in many vertical annular channel applications which may deteriorate the heat dissipation because of change from concentric to eccentric configurations.

Exact and approximate solutions of the heat conduction equation in eccentric cylindrical annuli with or without heat generation are available in the literature for three types of boundary conditions. Using bipolar coordinates, El-Saden [4] solved analytically the steady-state conduction equation in infinitely long eccentric cylindrical annuli with uniform heat generation rate when boundaries subjected to constant but different temperatures. His solution was based on the superposition of the solution of homogeneous equation, i.e. Laplace equation, and a particular solution. As long as a particular solution could be found El-Saden's solution is applicable to the conduction in eccentric cylindrical annuli with

* Corresponding author.
E-mail address: yilmazer@hacettepe.edu.tr (A. Yılmazer).

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Nomenclature

a	bipolar coordinates parameter, m
Bi	Biot number ($Bi = hr_o/k$)
d_1, d_2	distances to the foci, m
e	eccentricity, m
F_1, F_2	foci
g_ℓ	radial part of the Green's function
G	Green's function
h_ξ	scale factor (Eq. (6))
h_θ	scale factor (Eq. (6))
$\overline{h}_\xi$	normalized scale factor ($\overline{h}_\xi = h_\xi/a$)
$\overline{h}_\theta$	normalized scale factor ($\overline{h}_\theta = h_\theta/a$)
$I_{l,m}$	integral (Eqs. (30) and (B11))
k	thermal conductivity, W/(m K)
n	unit normal
$\dot{q}$	volumetric heat generation rate, W/m^3
Q	dimensionless heat generation ($Q = \dot{q}a^2/[k(T_i - T_\infty)]$)
$\overline{Q}$	dimensionless heat transfer rate
$\tilde{Q}(\xi,\theta)$	Eq. (25)
r	radius of sphere, m
S	boundary surface
T	temperature, K

Greek symbols

δ	Dirac's delta function, dimensionless eccentricity ($\delta = (e/r_i)/(r_o/r_i - 1)$)
∇^2	Laplacian
θ	coordinate
ξ	coordinate
Ψ	dimensionless temperature ($\Psi = (T - T_\infty)/(T_i - T_\infty)$)
ψ_ℓ	ℓth moment of the outer surface temperature, Eq. (26)

Subscript

i	inner
∞	ambient
o	outer
$<$	the smaller of the variables
$>$	the larger of the variables

Superscript

$'$	source coordinate (computational domain)
$-$	normalized

any combination of boundary conditions imposed on the two boundary surfaces. For uniform heat generation rate, El-Saden's formulation could be extended to any combination of boundary conditions on two boundary surfaces as El-Shaarawi and Mokheimer [3] demonstrated this for third kind boundary conditions. However, it may not be possible to propose a particular solution in general especially when source distribution is space dependent. Eckert and Drake [5] solved the same problem approximately by the superposition of the infinite line heat source and sink solutions. DeFelice and Bau [6] obtained an exact solution when convective boundary conditions are imposed on both boundary surfaces without heat generation. There are also studies which treat the conduction problems in eccentric cylindrical annuli numerically [7,8].

In a recent work by Moharana and Das [9], the conduction equation was solved approximately in an eccentric cylindrical annulus without heat generation when the inner surface kept at a constant temperature and the outer surface subjected to the convection by three different techniques: perturbation method, boundary collocation method, and sector method. They successfully applied these three approximate methods to the problem and obtained results both self-consistent and consistent with the literature. It is noted in their work that "heat transfer through an eccentric annulus with isothermal inner surface and convective outer surface cannot be solved analytically. Aziz [10] discussed this issue in detail". It is also stated in their work that a simple analytical expression cannot be obtained for the heat transfer through an eccentric annulus with isothermal inner surface and convective outer boundary. These statements and the literature search motivated the present work. A careful literature work revealed that there is no analytical work applicable in general which solves the heat conduction equation through a heat generated eccentric annulus with convective boundary condition(s). The existing analytical solutions as in Ref. [3] are limited to the constant heat generation case or without heat generation case as in Ref. [6]. In short, the desire to obtain an analytical solution for the case of any functional form of space dependency in heat generation when one or two boundaries subjected to convection is the main motivation of the present work.

To solve the conduction equation for an eccentric cylindrical annulus with heat generation, the first method that comes to mind is the method of eigenfunction expansion. Helmholtz equation for eccentric cylindrical geometry has been solved using different approaches, as for example conformal transformations mapping the eccentric annulus onto a concentric one. Since the transformed two dimensional Helmholtz equation is not conformally invariant, the transformed equation has the coordinate dependent coefficients and to be solved numerically [11]. Another approach is to express the solution in two different polar coordinates whose origins are the centers of cylinders forming the cylindrical annulus [12]. As each boundary surface fits to only one of the two polar coordinates systems used, the corresponding solutions satisfy one of the two boundary conditions exactly. Then, the two solutions are expressed in one of the coordinates system via translational properties of the eigenfunctions. This yields to relate unknown coefficients of the eigenfunctions of the two series solutions. The existence of the solution for the linear system of the unknown expansion coefficients results in a determinantal equation corresponding to a transcendental equation for the unknown eigenvalues. To solve this equation for quite a big number of eigenvalues is a formidable task in most practical cases. Even it is solved the utilization of the solution is still not very appropriate especially for the computations involving integration over whole domain.

In this study, the heat conduction equation for a heat generated eccentric annulus with isothermal inner surface and convective outer boundary is analytically solved. To do this, we treated the problem in bipolar coordinates system in which two boundary surfaces could be represented only with a single coordinate variable. Eigenfunction expansion method fails in analyticity since the Helmholtz equation is not separable or R-separable in bipolar coordinates [13]. Fortunately, Laplace equation is separable in bipolar coordinates which leads us to use Green's function method. With the presented method, the problem of both boundary surfaces subjected to the convection could easily be solved by the superposition of the presented solution and its counterpart obtainable via interchanging boundaries. Thus, the method involves the solutions for any combination of the first and third type