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Size effect in through-thickness conductivity of heterogeneous plates

H. Le Quang [a,*], D.C. Pham [b], G. Bonnet [a]

[a] Université Paris-Est, Laboratoire de Modélisation et Simulation Multi Echelle, UMR 8208 CNRS, 5 Bd Descartes, F-77454 Marne-la-Vallée Cedex 2, France
[b] Institute of Mechanics – Vietnam Academy of Science and Technology (VAST), 264 Doi Can, Hanoi, Viet Nam

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ABSTRACT

In this work, the size effect on the effective through-thickness conductivity of heterogeneous plates expressed in second-order Hashin–Shtrikman bounds and a third-order correlation approximation is studied. By taking into account the homogeneous temperature boundary conditions, the exact Green operator for the plate is first established. Then, the respective bounds and correlation approximation are constructed. With the help of the method based on the fast Fourier transform (FFT), the bounds and correlation approximation for the effective through-thickness conductivity are computed for the plates reinforced or weaken randomly either by spherical particles or unidirectional fibers. The numerical results show that the size effect of the effective through-thickness conductivity is more significant than the one of the effective in-plane conductivity.

1. Introduction

Macroscopic (effective) properties of randomly inhomogeneous materials are generally hard to be determined theoretically, because of the random irregular nature of their microgeometry. Hence, variational approaches have been developed to construct upper and lower bounds on the possible values of the effective properties of the composites, which may involve multi-point correlation parameters describing the microstructure of a composite, besides the properties of the component materials [1–8]. n-point correlation functions are related to the probability of finding n points in certain relative arrangement, i.e. in the spaces of certain components. One-point correlation information about a particular component is just its volume proportion. High-order correlation information about a composite is hard to collect and to include into an estimate, hence one has to restrict oneself to the lowest-order correlation functions, in particular, the two-point and three-point ones, the values of which have been tabulated for a number of practical microgeometries (see e.g. Ref. [7]).

Alternatively, effective medium approximation schemes have been developed to estimate the effective properties of the composites [7,9–13]. Refined approximations incorporate correlation information about composites' microgeometry [2,7,14–18]. Developed upon the work of Brown [19], Sen and Torquato [15] derived strong contrast expansions for the effective conductivity tensor of macroscopically anisotropic two-phase media. Pham and Torquato [18] extended further the approach to the n-phase composites. From the expansions, they proposed the three-point correlation approximation for the effective conductivity of isotropic composites that, in the case of two-phase materials, agrees well with numerical results for a number of periodic and random composites, even when the contrast between the phases is infinite and their volume proportions are near percolation thresholds. The simple approximation reduces to the well-known Maxwell and self-consistent ones for the respective asymmetric matrix-inclusion composites and symmetric cell mixtures, and it obeys second-order Hashin–Shtrikman as well as third-order three-point correlation bounds over all the ranges of parameters.

One is interested in the effect of restricted domains when the sizes of heterogeneities are no more negligible compared with a characteristic size of the domain, leading to a well-defined size effect. This size effect has been studied for plates, because the geometry allows to extend precisely the results obtained in infinite domains; this size effect appears when the size of heterogeneities has the order of the thickness of the plate. For plate problems, first order and second order bounds were extended in the case of elastic properties [20–23]. In Ref. [24], we extended Pham–Torquato three-point correlation approximation and second order Hashin–Shtrikman bounds, accounting for the size effect and insulation boundary condition, on the effective in-plane conductivity of heterogeneous plates. In this work we continue to develop the

* Corresponding author. Tel.: +33 (0) 1 60 95 77 97; fax: +33 (0) 1 60 95 77 99.
E-mail addresses: hung.lequang@univ-paris-est.fr, hung.lequang@univ-mlv.fr (H. Le Quang).

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approach in order to study the size effect on through-thickness conductivity of heterogeneous plates.

In Section 2, a Green operator for the heterogeneous plate with homogeneous temperature boundary condition is constructed. The Hashin–Shtrikman-type bounds on the through-thickness effective conductivity are extended in Section 3. Section 4 develops the correlation approximation for the through-thickness effective conductivity. Numerical applications are provided in Section 5, followed by conclusions.

2. A Green operator for heterogeneous plate

In a three-dimensional Euclidean space $\mathbb{R}^3$, let us consider a heterogeneous plate consisting of spherical or unidirectional inhomogeneities embedded in a matrix phase. The matrix, referred to as phase 1, and inhomogeneity, denoted by phase 2, are assumed to be individually homogeneous and have the linear thermal-conduction behavior described by a local isotropic Fourier's law

$$\mathbf{q}(\mathbf{x}) = c(\mathbf{x})\mathbf{E}(\mathbf{x}). \tag{1}$$

Here the local intensity field $\mathbf{E}(\mathbf{x})$ is the opposite of the gradient of temperature field $T(\mathbf{x})$

$$\mathbf{E}(\mathbf{x}) = -\nabla T(\mathbf{x}), \tag{2}$$

while local heat flux vector field $\mathbf{q}$ at position $\mathbf{x}$ must verify the following energy conservation equation

$$\nabla\cdot\mathbf{q}(\mathbf{x}) = 0 \tag{3}$$

in the case of stationary thermal conduction without heat source. The local scalar conductivity at position $\mathbf{x}$ is expressible as

$$c(\mathbf{x}) = \sum_{\alpha=1}^{2} c_\alpha \mathcal{I}^{(\alpha)}(\mathbf{x}), \tag{4}$$

where $\mathcal{I}^{(\alpha)}(\mathrm{x})$ is the indicator function of phase α ($\alpha = 1$ or 2) which is defined in such a way that $\mathcal{I}^{(\alpha)}(\mathrm{x}) = 1$ if $\mathbf{x}$ in phase α and otherwise $\mathcal{I}^{(\alpha)}(\mathrm{x})=0$. For statistically homogeneous media, $\langle\mathcal{I}^{(\alpha)}(\mathrm{x})\rangle = v_\alpha$, where angular brackets denote an ensemble average.

For later use, we denote by Z the three-dimensional domain occupied by a simple or representative volume element (RVE) of the heterogeneous plate. More precisely, the latter can be defined by

$$Z = \left\{\mathbf{x}\in\mathbf{R}^3,\ \mathbf{x} = (x_1,x_2,x_3), x_\alpha\in\left]-\frac{l_\alpha}{2},\frac{l_\alpha}{2}\right[, x_3\in\left]-\frac{t}{2},\frac{t}{2}\right[\right\}, \tag{5}$$

where $\alpha = 1$ or 2; l_1, l_2 and t are the length, width and thickness of Z, respectively. We designate by $\omega =]-l_1/2,l_1/2\ [\times]\ -l_2/2,l_2/2[$ the middle surface of Z and by $\partial\omega$ the boundary of ω. The lateral boundary ∂Z_l of Z is defined by $\partial Z_l = \partial\omega \times]-t/2,\ t/2[$. The top and bottom surfaces $\partial Z^\pm$ of Z are $\partial Z^\pm = \omega \times (\pm t/2)$ (see Fig. 1).

In order to determine the effective through-thickness conductivity of the heterogeneous plate, let Z be subjected to the zero reference temperature ($T = 0$) on the bottom surface ∂Z^- and a constant temperature ($T = T^0$) on the top surface ∂Z^+. In addition, a periodic boundary condition is imposed on the lateral boundary ∂Z_l of Z. The determination of the Hashin–Shtrikman-type bounds (Section 3) as well as the correlation approximation (Section 4) for the through-thickness conductivity of composite plates with possible finite size (thickness) effect needs first to construct the Green operator for the heterogeneous plates with zero temperature boundary conditions. This conduction problem defined on Z can be expressed in the following form

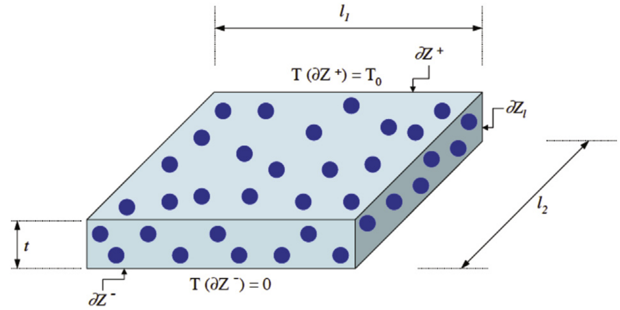

Fig. 1. Description of a representative volume element (RVE) of heterogeneous plates.

$$\begin{cases} \nabla\cdot\mathbf{q}(\mathbf{x}) = 0 \quad \text{in } Z, \\ \mathbf{q}(\mathbf{x}) = c(\mathbf{x})\mathbf{E}(\mathbf{x}), \quad \mathbf{E}(\mathbf{x}) = \mathbf{E}^0 + \mathbf{E}(\mathbf{x}) \quad \text{in } Z, \\ \mathbf{E}(\mathbf{x}) = -\nabla T^{\mathrm{per}}(\mathbf{x}) \quad \text{in } Z, \\ T^{\mathrm{per}}(\mathbf{x}) \text{ periodic on } \partial Z_l, \\ \mathbf{q}(\mathbf{x})\cdot\mathbf{n} \text{ anti} - \text{periodic on } \partial Z_l, \\ T^{\mathrm{per}}(\mathbf{x}) = 0 \text{ on } \partial Z^\pm. \end{cases} \tag{6}$$

Here, $\mathbf{E}^0$ being a constant macroscopic gradient field is chosen in such a way that the non-zero through-thickness component is $E_3^0 = T^0/t$ and $E_1^0 = E_2^0 = 0$. By introducing the reference medium of conductivity c_0, the conduction problem (6) is equivalent to

$$\begin{cases} \nabla\cdot\mathbf{q}(\mathbf{x}) = 0 \quad \text{in } Z, \\ \mathbf{q}(\mathbf{x}) = c_0\mathbf{E}(\mathbf{x}) + \mathbf{P}(\mathbf{x}), \quad \mathbf{E}(\mathbf{x}) = \mathbf{E}^0 + \mathbf{E}(\mathbf{x}) \quad \text{in } Z, \\ \mathbf{E}(\mathbf{x}) = -\nabla T^{\mathrm{per}}(\mathbf{x}) \quad \text{in } Z, \\ T^{\mathrm{per}}(\mathbf{x}) \text{ periodic on } \partial Z_l, \\ \mathbf{q}(\mathbf{x})\cdot\mathbf{n} \text{ anti} - \text{periodic on } \partial Z_l, \\ T^{\mathrm{per}}(\mathbf{x}) = 0 \text{ on } \partial Z^\pm. \end{cases} \tag{7}$$

In (7), the polarization $\mathbf{p}(\mathbf{x})$ is given by $\mathbf{p}(\mathbf{x}) = [c(\mathbf{x}) - c_0]\mathbf{E}(\mathbf{x})$. To solve the problem (7), we decompose (7) into two auxiliary problems. The first one is obtained from (7) with $\mathbf{p}(\mathbf{x}) = 0$ while the second one is provided from (7) by setting $\mathbf{E}^0 = 0$.

It is clear that the first auxiliary problem has a trivial solution $\mathbf{E}(\mathbf{x}) = \mathbf{E}^0$, i.e. $\mathbf{e}(\mathbf{x}) = 0$. For the second auxiliary problem, due to the linearity of the local constitutive laws, the intensity solution field $\mathbf{E}(\mathbf{x})$ is related linearly to the periodic polarization field $\mathbf{p}(\mathbf{x})$ by

$$\mathbf{E}(\mathbf{x}) = -\frac{1}{|Z|}\int_Z \mathbf{G}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{P}(\mathbf{x}')d\mathbf{x}', \tag{8}$$

where $|Z|$ is the volume of Z and $\mathbf{G}$ is the Green operator with zero temperature boundary conditions. Moreover, this intensity solution field $\mathbf{E}(\mathbf{x})$ can be decomposed into 2 parts as $\mathbf{E}(\mathbf{x}) = \mathbf{E}^p(\mathbf{x}) + \mathbf{E}^c(\mathbf{x})$. The first part $\mathbf{E}^p(\mathbf{x})$ corresponds to the intensity solution field of the periodic boundary value problem in which the domain Z medium with periodic boundary conditions on ∂Z_l and $\partial Z^\pm$ is undergone by the periodic polarization field $\mathbf{p}(\mathbf{x})$. More precisely,

$$\begin{cases} \nabla\cdot\mathbf{q}^p(\mathbf{x}) = 0 \quad \text{in } Z, \\ \mathbf{q}^p(\mathbf{x}) = c_0\mathbf{E}^p(\mathbf{x}) + \mathbf{P}(\mathbf{x}) \quad \text{in } Z, \\ \mathbf{E}^p(\mathbf{x}) = -\nabla T^p(\mathbf{x}) \quad \text{in } Z, \\ T^p(\mathbf{x}) \text{ periodic on } \partial Z_l, \\ \mathbf{q}^p(\mathbf{x})\cdot\mathbf{n} \text{ anti} - \text{periodic on } \partial Z_l, \text{ and } \partial Z^\pm. \end{cases} \tag{9}$$

By introducing the Green operator $\mathbf{G}^p$ with periodic boundary conditions, the intensity solution field $\mathbf{E}^p(\mathbf{x})$ takes in the following form

$$\mathbf{E}^p(\mathbf{x}) = -\frac{1}{|Z|}\int_Z \mathbf{G}^p(\mathbf{x}-\mathbf{x}')\cdot\mathbf{P}(\mathbf{x}')d\mathbf{x}'. \tag{10}$$