International Journal of Thermal Sciences 79 (2014) 176–182

Contents lists available at ScienceDirect

International Journal of Thermal Sciences

journal homepage: www.elsevier.com/locate/ijts

Unstable buoyant flow in an inclined porous layer with an internal heat source

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ARTICLE INFO

Article history:
Received 12 August 2013
Received in revised form
17 December 2013
Accepted 3 January 2014
Available online 14 February 2014

Keywords:
Porous medium
Darcy's law
Convective instability
Internal heating
Buoyant flow
Normal modes

ABSTRACT

The buoyant flow with zero average velocity, namely free convection, in an inclined porous layer is studied. The heating is supplied by an internal volumetric source with a uniform distribution. The boundaries are either isothermal at the same temperature, or the lower one adiabatic and the upper one isothermal. The stability to small-amplitude perturbations is analysed for three-dimensional normal modes. It is proved that the longitudinal rolls, *viz.* normal modes with wave vector perpendicular to the basic flow, are the most unstable modes. It is also shown that neutrally stable transverse modes may grow in time if the inclination angle of the layer to the horizontal is smaller than a threshold value. The threshold angle depends on the imposed boundary conditions, isothermal/isothermal or adiabatic/isothermal. When the threshold angle is approached from below, the neutral stability curves assume a closed-loop shape, they gradually shrink their size and eventually collapse to a point.

1. Introduction

A vast amount of literature has been published on the onset of convection, due to heating from below, in a layer of saturated porous medium (the Horton–Rogers–Lapwood problem [1,2]). Comparatively little work has been done on the corresponding problem where the convection is induced by uniform internal heating (see the survey in Section 6.11.2 of Nield and Bejan [3]). In fact the case of internal heating is particularly interesting, because not only is the basic thermal gradient not constant but it changes sign within the layer, and convection is concentrated in that portion where that upwards vertical gradient is negative. This asymmetry about the mid-layer level has some interesting results, especially where heterogeneity is involved (see, for example, Nield and Kuznetsov [4]). Also of particular interest are those situations where the basic flow is not zero, but rather flow is parallel to the layer boundaries. One such situation, where there is a basic Hadley type circulation produced by an oblique applied temperature gradient, has been studied by Parthiban and Patil [5]. However, with one exception, we are not aware of any published study of the related problem where the layer is inclined to the horizontal, and the aim of this paper is to fill that gap. The exception is a short conference paper by Storesletten and Rees [6]. These authors considered only the case of constant-temperature boundary conditions, and analysed the instability to a special kind of disturbance mode, *viz.* the longitudinal rolls. In fact a comparison between this case and that where one or both of the boundaries is held at constant flux is of interest. In the case of bottom heating the effect of this change is readily predictable. The basic solution is not changed. The replacement of specified temperature to specified temperature gradient means a relaxation of a boundary condition and hence a reduction of the eigenvalue (the Rayleigh number) in the perturbation differential equation system. In the case of internal heating a prediction is less readily made because the basic solution is dependent on the thermal boundary conditions.

The case of an inclined layer subject to heating of the lower boundary but with no internal heating has been extensively studied, and this work has been surveyed in Section 7.8 of Nield and Bejan [3]. In particular, the linear instability of the Darcy–Hadley flow in an inclined layer was studied by Barletta and Rees [7], and this work has guided the present investigation. We mention that Barletta and Rees [7] developed their analysis in the wake of previous papers dealing with the convective instability in inclined porous layers [8–14].

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http://dx.doi.org/10.1016/j.ijthermalsci.2014.01.002

Nomenclature

$\widehat{\boldsymbol{e}}_x, \widehat{\boldsymbol{e}}_y, \widehat{\boldsymbol{e}}_z$	unit vectors along the (x, y, z)-axes
$f(z)$, $h(z)$	dimensionless amplitude functions, Eq. (12)
$\boldsymbol{g}$, g	gravitational acceleration, modulus of $\boldsymbol{g}$
H	layer thickness
k	dimensionless wave number, $(k_x^2 + k_y^2)^{1/2}$
(k_x, k_y)	components of the dimensionless wave vector, Eq. (12)
K	permeability
Q_0	thermal power generated per unit volume
R	internal Darcy–Rayleigh number, Eq. (4)
S	transformed Darcy–Rayleigh number, Eq. (14)
t	dimensionless time, Eq. (1)
T	dimensionless temperature, Eq. (1)
T_0	wall temperature
$\boldsymbol{u}$	dimensionless velocity, (u, v, w), Eq. (1)
$\boldsymbol{U}$	dimensionless velocity perturbation, Eq. (7)
W	z-component of $\boldsymbol{U}$
(x, y, z)	dimensionless Cartesian coordinates, Eq. (1)

Greek symbols

α_m	average thermal diffusivity
β	thermal expansion coefficient
ε	dimensionless perturbation parameter, Eq. (7)
η	dimensionless switch parameter: either 0 or 1
Θ	dimensionless temperature perturbation, Eq. (7)
κ_m	average thermal conductivity
ν	kinematic viscosity
ξ_1, ξ_2	dimensionless parameters, Eq. (17)
σ	heat capacity ratio
ϕ	inclination angle of the layer to the horizontal
φ	transformed inclination angle, Eq. (14)
ω	dimensionless angular frequency, Eq. (12)

Subscripts

b	basic solution
c	critical value

2. Governing equations

Let us consider a plane porous layer with thickness H inclined an angle ϕ to the horizontal. We model the saturated porous medium as homogeneous, isotropic and subject to uniform internal heating with power per unit volume $Q_0 > 0$. The boundary walls are assumed to be impermeable, with the upper wall isothermal at a temperature T_0. The lower wall is either isothermal at temperature T_0, or adiabatic. A sketch of the porous layer and of the thermal boundary conditions is displayed in Fig. 1.

The dimensionless coordinates (x, y, z), time t, temperature T, and velocity $\boldsymbol{u} = (u, v, w)$ are made dimensionless through the scalings

$$\begin{gathered}(x,y,z)\frac{1}{H}\to(x,y,z), \quad t\frac{\alpha_m}{\sigma H^2}\to t, \quad (T-T_0)\frac{g\beta K H}{\alpha_m \nu}\to T,\\ \boldsymbol{u}\frac{H}{\alpha_m} = (u,v,w)\frac{H}{\alpha_m}\to(u,v,w) = \boldsymbol{u},\end{gathered} \tag{1}$$

where α_m is the average thermal diffusivity, ν is the kinematic viscosity, β is the thermal expansion coefficient, g is the modulus of the gravitational acceleration $\boldsymbol{g}$, K is the permeability, and σ is the ratio between the average volumetric heat capacity of the saturated medium and that of the saturating fluid.

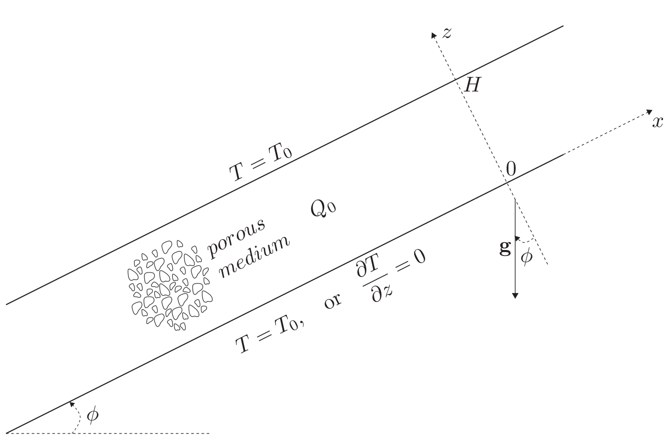

Fig. 1. Sketch of the inclined porous layer.

According to the Oberbeck–Boussinesq approximation and to Darcy's law we can write the mass, momentum and energy balance equations as

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2a}$$

$$\nabla \times \boldsymbol{u} = \nabla \times (T\sin\phi\, \widehat{\boldsymbol{e}}_x + T\cos\phi\, \widehat{\boldsymbol{e}}_z), \tag{2b}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T = \nabla^2 T + R. \tag{2c}$$

The local momentum balance equation (2b) is formulated by applying the curl operator to both sides of Darcy's law, so that the pressure gradient term gives no contribution. The unit vectors along the coordinate axes are denoted as $(\widehat{\boldsymbol{e}}_x, \widehat{\boldsymbol{e}}_y, \widehat{\boldsymbol{e}}_z)$.

The boundary conditions are expressed as

$$\begin{aligned} z = 0: &\quad w = 0, \quad \eta T - (1-\eta)\frac{\partial T}{\partial z} = 0,\\ z = 1: &\quad w = 0, \quad T = 0,\end{aligned} \tag{3}$$

where η is a switch variable that can be either equal to 1 for an isothermal lower wall, or equal to 0 for an adiabatic lower wall.

In Eq. (2c), we introduced the internal Darcy–Rayleigh number R defined as

$$R = \frac{g\beta K H^3 Q_0}{\alpha_m \nu \kappa_m}, \tag{4}$$

where κ_m is the average thermal conductivity of the porous medium.

3. Basic solution

Equations (2) and (3) can be solved analytically under stationary conditions, by assuming that the temperature field depends only on the z-coordinate, and considering the velocity field as parallel and directed along the x-axis. Thus, we obtain

$$\boldsymbol{u}_b = -\frac{R}{12}\left(3\eta - 2 - 6\eta z + 6z^2\right)\sin\phi\, \widehat{\boldsymbol{e}}_x, \tag{5a}$$