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A numerical-analytical solution to the mixed boundary-value problem of the heat-conduction theory for arbitrarily inhomogeneous coatings

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ABSTRACT

This paper presents a technique for the solution of mixed-type axisymmetric heat-conduction problems for an inhomogeneous half-space consisting of an arbitrarily inhomogeneous coating and a homogeneous substrate. The boundary surface of the half-space is thermally insulated with except for a circular area, where either the heat flux or temperature is given. For the case of mixed-type boundary conditions, the bilateral asymptotic method was employed in order to construct the numerical-analytical solutions to the arising dual integral equations, whose kernels have been operated by means of the method of modeling functions. Numerical analysis of the temperature and transversal heat flux in the inhomogeneous half-space has been performed for different variations of the heat-conduction coefficient within the thickness of the inhomogeneous coating.

1. Introduction

The analysis of temperature fields and heat fluxes distributed in solids with inhomogeneous coatings, which undergo the mixed-type thermal exposure on the surface, presents a subject of concern to the thermal science specialists from both theoretical (as a good field for the development of efficient mathematical models and stable analytical and numerical methods) and applied (with concern to the great number of practical implementations of inhomogeneous coatings) standpoints. Such problems arise, for instance, in the process of the laser irradiation in opaque bodies, solar batteries, thermal vessels with protective coatings, as well as a great number of thermally-affected industrial-, power-, and aircraft-moduli, etc. In a large measure, the topicality of such problems is also induced by the wide application of structure members with protective functionally-graded coatings.

The functionally-graded materials (FGM) are a class of multi-phase composites combining two or more phase-materials with contrast properties. One of the key features of FGM is the continuous (or "almost continuous") volumetric variation of the phase-constituents percentage from one surface to another [1,2]. Such variation effects in an inhomogeneous microstructure with mechanical and thermophysical properties of continuous variation within the solid. The concept and term of FGM originated in Japan in late 1980s within the framework of the program on development of heat-resistant materials [3,4]. However, the research aimed on materials with graded properties, have been initiated much earlier [5]. The basic advantage of FGM is the ensuring of effective thermal barrier along with the high-tensile mechanical performance achieved by combination of metal and ceramic2 constituents [6–8]. The continuous variation in the material properties of FGM, which are used as an intermediate layer between metal and ceramic parts of a structure, allows for the significant reduction of the residual thermal stresses on the interface [9–11]. With this concern, the development of efficient methods for analysis of thermal fields and fluxes in FGM, as a stage of the FGM optimization, appears to be an important problem of modern thermal engineering.

The analytical and numerical solutions of the basic problems of mathematical physics (particularly, the heat-conduction, elasticity and thermoelasticity ones) for continuously inhomogeneous solids present a challenge due to the fact that the governing equations of

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such problems contain variable coefficients, which, in general, are arbitrary functions of spatial coordinates [12]. This fact complicates the separation of variables in the mentioned equations as well as minimizes the applicability of classical methods to the solution of relevant initial-boundary-value problems. To overcome these difficulties, a number of simplifying assumptions and approximate methods have been developed for particular practical cases of inhomogeneity.

Among the latter methods, the dominant one implies the material properties in the form of certain elementary functions (e.g., the linear, polynomial, exponential, etc.) in order to separate the variables in the governing equations and, on the other hand, to construct a comparatively simple fundamental solutions. Due to the latter advantage, particularly, this method appeared to be quite popular so that it is nearly impossible to present an exhaustive list of papers devoted this method in the case of material properties depending on one of the spatial directions (which is, in the many cases, perpendicular to the boundary of a considered solid). The reviews of particular results obtained for the dependences of a special kind are given, e.g., in [13–15]. The number of papers considering dependences of material properties on more than one coordinate is quite limited. A heat-conduction problem for a half-space, whose heat-conduction coefficient depends exponentially on all three Cartesian coordinates, has been attempted by Wang et al. [16] by means of the Green-function method. Two- and three-dimensional solutions to the stationary heat-conduction problems for an anisotropic solid with exponential variation of the heat-conduction coefficient within an arbitrarily-orientated direction were suggested by Berger et al. [17]. Despite the substantial idealization, the representation of material properties by elementary functions allowed for the construction of a number of benchmark solutions for continuously-inhomogeneous solids. On the other hand, application of this method is not always reliable. First of all, this method does not provide the sufficient generality of a solution and, thus, when changing the type of inhomogeneity, it is necessary to resolve the problem by construction, in general, a new system of fundamental solutions for the governing equations. I addition, the representation of material properties by monotonic functions for the solids, which are modeled by unbounded regions, can be a cause of unfeasible results due to fact that such representation brings the material properties beyond the modeling restrictions (see, e.g., [18]).

The mentioned drawbacks can be obviated by application of the method, whose basic idea rests upon the representation of a solid, exhibiting continuous variation of its properties in one of the spatial directions, by a multilayer composite of the same shape consisting of a number of perfectly connected homogeneous layers so that the entire assembly of dissimilar material properties approximates the original distribution of inhomogeneity [19–21]. By constructing solutions to the problems for each homogeneous layer, the solution for entire solid can be finally approximated via tailoring the obtained solutions by making use of the interlayer contact conditions. This method allows for the analysis of different dependences of the material properties by means of the same solution strategy. However, this method is not quite profitable for the analysis of high-gradient inhomogeneities, which is often concerned with a significant number of layers of lesser thickness. To avoid the other drawback arising within the framework of this method (the discontinuity of shearing stress on the interfaces between dissimilar homogeneous layers), a certain modification of the method can be employed. In order to achieve the continuity of the material properties across the interlayer interfaces, this modification is concerned with representation of the material properties of each single layer by elementary functions, e.g., the linear or exponential ones [22,23]. A numerical solution for the case of smooth and nonsmooth material properties on the interfaces of exponentially-graded layers has been performed by Marin [24] by making use of the zero-order Tikhonov's functional. It is obvious that the efficient implementation of this method is based on the solution techniques developed for multilayer solids [25–28], which also present a class of inhomogeneous materials.

An important example of inhomogeneous solids is the “substrate-coating” systems, where the coating can be regarded as either homogeneous or inhomogeneous (or multilayer) by causing continuity or discontinuity of the material properties on the interface. A technique for the approximate analysis of heat-conduction problems in solids with thin multilayer coatings was suggested by Shevchuk [29]. By taking into consideration the fact that coatings are thin, their effect has been modeled by introducing the generalized boundary conditions. A solution to the three-dimensional thermoelasticity problem for a half-space with inhomogeneous coating, whose Young's modulus is a power function of depth, has been obtained in [30] and analyzed by its comparison to a multilayer model. A solution to the plane problem for multilayer and gradient coatings resting upon a metallic substrate has been constructed by Lee & Erdogan [31]. By comparison of the obtained results, the decrement estimation for singular stresses on the interfaces has been performed.

A special kind of inhomogeneity can be induced by the dependence of their properties on non-uniform temperature distributions [32]. I this case, the governing heat-conduction equation is, in general, non-linear due to the dependence of its coefficients on the quested-for function. Solutions of such problems can be obtained numerically by means of the linearization techniques [33,34]. Moreover, the nonlinearity of the heat-conduction equation can be supplemented by the nonlinearity of certain boundary conditions, for instance, the conditions of convective heat exchange. A numerical-analytical method for analysis of such problems has been suggested by Kushnir & Popovych [35].

An efficient approach to the analysis of arbitrarily inhomogeneous and thermosensitive solids is based on the implementation of the direct integration method [36] along with the consequent reduction of the original problems to solution of the integral equations [37].

The case of mixed-type boundary conditions (for instance, when the Dirichlet-type boundary conditions are given on a part of the boundary, meanwhile the Neumann-type ones are imposed on the remaining part) presents a challenge for both analytical and numerical modes of attack, even for homogeneous materials [38]. The basic methods for the solution of mixed-type boundary-value problems are presented in [39]. In the case of inhomogeneous materials, the mixed-type boundary-value problems were usually attempted by means of numerical methods, for instance, the boundary-element or boundary-integral method [40,41]. Gray et al. [42] used the method of Green function for the solution of a steady-state heat-conduction problem for exponentially-inhomogeneous solid in the two- and three-dimensional formulation. The solution has been realized by means of the algorithm of boundary elements along with the non-symmetric Galerkin method used for the approximation of governing expressions and the six-node isoparametric triangular element for the interpolation of the boundary and boundary conditions. In [43], this approach has been extended to the case of non-stationary problems by making use of the Laplace transformation along with a numerical procedure for its inversion.

A great number of mixed problems arise from the contact thermomechanics [44], where the mutual mechanical and thermal interaction of contacting solids within a limited part of their surface calls for the formulation of mixed-type boundary conditions. Kulchytsky-Zhyhailo & Rogowski [45] considered a problem on the