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Asymmetrical collision of thermal waves in thin films: An analytical solution

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ABSTRACT

The purpose of this study is to examine thermal wave phenomena in a thin finite film subjected to non-homogeneous boundary conditions. The Cattaneo–Vernotte (C–V) heat conduction model is solved using the superposition principle in conjunction with the solution structure theorems. For comparison purposes, the diffusion model is also solved to demonstrate the flexibility in the technique as well as to show the differences in the results. It is recognized that the solution structure theorems are suitable for homogeneous systems only. However, by performing a functional transformation, the original non-homogeneous partial differential equation governing the physical problem can be cast into a new form with homogeneous boundary conditions such that it can be solved directly with the solution structure theorems. In this study, details of this process will be examined and explored for achieving solutions in such systems. The methodology provides a convenient technique for the solution of the diffusion and C–V heat conduction equations with non-homogeneous boundary conditions.

1. Introduction

Thermal energy transport in solids is conducted by electron particles and phonon quanta. Its traveling speed should not be greater than that of light. This implies that the impact of one localized area subjected to a sudden change of temperature or heat flux exposure should not be instantaneously sensed at any other locations within the medium. Most analysis of heat transfer in solids has been based on the macroscopic diffusion theory, or Fourier law, and is widely accepted in the scientific and engineering community. The empirical law of diffusive heat conduction governs thermal energy transport in solids, stating that the rate of heat flow in a given direction is proportional to the area normal to the direction of travel and to the temperature gradient in the same direction. The classical Fourier parabolic heat equation assumes that thermal energy travels within the solid medium at a non-physical infinite speed. This is a valid assumption for typical applications but breaks down in situations that include low temperature conditions or engineering applications such as material processing [1] with high-power for short duration, e.g., laser welding, explosive bonding, electrical discharge machining, and heating and cooling of micro-electronic elements, etc. To that end, a great deal of academic interest since the 1950s has been devoted towards seeking a thermal model that can predict a finite speed of propagation [2–6]. One of the most popular models is the commonly known Cattaneo–Vernotte (C–V) hyperbolic thermal model [3,4].

Thermal propagation in thin film subjected to a sudden temperature change on its surfaces was investigated by Tan and Yang [1]. The investigation was extended to study the wave nature of heat propagation in a thin film subjected to an asymmetrical temperature change on both sides [7]. Both studies presented analytical expressions for the temperature and heat flux distributions and numerical results for the time history of heat transfer behaviors using the method of separation of variables. Results revealed that transient heat conduction traveled in a wave form and attenuated within the medium. The studies contradicted results based on the Fourier law in macroscale heat conduction. Torii and Yang [8] also studied the heat transfer mechanisms in a thin film with symmetrical laser heat source impingement on its boundaries. Temperature solutions were obtained by using MacCormack's predictor–corrector scheme [9]. The same problem was solved analytically by the method of Laplace transforms by Lewandowska and Malinowski [10]. Results have demonstrated that temperature overshoot could occur in thin films within a short period of time but does not appear in thicker films.

In this study, a physical problem similar to the one considered by Tan and Yang [1,7] will be studied. A thin film subjected to asymmetrical boundary conditions on both sides is examined. A

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Nomenclature

c	thermal wave speed
c_p	specific heat
$\mathcal{F}$	function
f	forcing function
f_1, f_3	boundary condition at $x = 0$
f_2, f_4	boundary condition at $x = 1$
f_r	referenced heat flux
g	dimensionless internal heat generation
$\mathscr{H}_1$	dimensionless initial condition
$\mathscr{H}_2$	dimensionless initial temperature rate of change
k	thermal conductivity
$\mathscr{M}$	coefficient, Eq. (9)
$\mathscr{N}$	coefficient, Eq. (9)
N	norm
n	index
q	heat flux
T	dimensionless temperature
t	dimensionless time
u	dimensionless temperature solution for the homogeneous problem
u_1	dimensionless temperature solution due to ψ function contribution
u_2	dimensionless temperature solution due to φ function contribution
u_3	dimensionless temperature solution due to f function contribution
w	auxiliary function
X	eigenfunction
x	dimensionless slab thickness

Greek symbols

α	thermal diffusivity
β_n	eigenvalue, Eq. (18c)
γ_n	eigenvalue, Eq. (23d)
ε	relative error
η_n	eigenvalue, Eq. (30d)
λ_n	eigenvalue, Eq. (27c)
ϖ_n	eigenvalue, Eq. (30d)
ζ	dummy variable for time
ξ	dummy variable for space
ρ	density
τ_{CV}	relaxation time
φ	initial condition function
ψ	initial temperature rate of change function

Superscript

*	dimensional quantity

non-Fourier heat conduction problem formulated using the Cattaneo–Vernotte (C–V) model with non-homogeneous boundary conditions is solved with the superposition principle [11] in conjunction with solution structure theorems [12]. This technique has been applied successfully for the study of hyperbolic heat conduction by Lam and Fong [13,14] for homogeneous problems. It is well known that the aforementioned analytical method is not suitable for such classes of non-homogeneous thermal problems. However, by performing a functional transformation, the original non-homogeneous partial differential equation governing the physical problem can be cast into a new form such that it consists of a homogeneous part and an additional auxiliary function. As a result, the modified homogeneous governing equation can then be solved with the solution structure theorems for temperatures inside a finite planar medium. The resulting temperature profile is obtained in the form of a series solution. The method is relatively simple and requires only a basic background in applied mathematics. The methodology provides a simple technique for the solution of the C–V heat conduction equation with non-homogeneous boundary conditions as compared to conventional methods, e.g., separation of variables and Laplace transforms. The present study will demonstrate the application of the method.

The outline of the paper is as follows. Section 2 presents the formulation of the non-homogeneous physical problem for both the diffusion and C–V heat conduction models. General non-homogeneous boundary and initial conditions are also presented. Section 3 outlines the solution method. The governing equation is recast into a homogeneous form through a transformation process with the aid of an auxiliary function such that the resulting modified partial differential equation can be directly solved for the temperature field by applying the superposition method and solution structure theorems. Section 4 presents the general temperature solution for both the diffusion and hyperbolic models. In Section 5 the temperature solution for a specific set of internal heat generation, boundary and initial conditions is presented for the physical problem under consideration. Section 6 performs the numerical simulation of the temperature profiles. Results are presented for both the diffusion and hyperbolic models. A summary of the study can be found in Section 7.

2. Physical problem and heat conduction models

In the present study, an isotropic slab confined to the region $0 \le x \le 1$ with uniform thickness and constant thermophysical properties, is assumed. Initially, the slab is at temperature $T(x,0) = \mathscr{H}_1(x)$, which is the prescribed spatial temperature distribution within the solid. For time $t > 0$, the boundary at $x = 0$ is maintained at a constant temperature. The boundary at $x = 1$ is either kept at a constant temperature or is subjected to an incident heat flux. In other words, the aforementioned conditions are either first type (prescribed temperature) or second type (prescribed heat flux) boundary conditions. Internal heat generation inside the medium is designated as $g(x,t)$. Temperature distributions within the slab will be obtained based on the diffusion and hyperbolic Cattaneo–Vernotte (C–V) thermal models. The governing equation, boundary and initial conditions governing the heat transfer process within the solid slab have been presented in detail by Lam and Fong [13,14]. They are summarized as follows by using the indicial notation for brevity.

2.1. Heat conduction models

2.1.1. Diffusion model

The parabolic Fourier law heat conduction equation specifies that the heat flux conducted across a solid is proportional to the temperature gradient taken in a direction normal to the material surface in question. Using indicial notations, the Fourier law is given as

$$q^* = -kT^*_{x^*} \quad (1)$$

The energy conservation law is given by

$$\rho c_p T^*_{t^*} = -q^*_{x^*} + g^* \quad (2)$$

Note that the subscripts t^* and x^* shown above represent the