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# Slip-flow heat transfer in microtubes with axial conduction and viscous dissipation – An extended Graetz problem

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ABSTRACT

This study is an extension of the Graetz problem to include the rarefaction effect, viscous dissipation term and axial conduction with constant-wall-heat-flux thermal boundary condition. The energy equation is solved analytically by using general eigenfunction expansion. The temperature distribution and the local Nusselt number are determined in terms of confluent hypergeometric functions. The effects of the rarefaction, axial conduction and viscous dissipation on the local Nusselt number are discussed in terms of dimensionless parameters such as the Knudsen number, Peclet number and Brinkman number.


## 1. Introduction

The Graetz problem, which is the problem of hydrodynamically-developed, thermally developing laminar flow of an incompressible fluid inside a tube neglecting the axial conduction and viscous dissipation, was solved analytically by Graetz [1,2] and Nusselt [3] more than a century ago. Many studies extended the Graetz problem to include additional effects (such as the axial conduction and viscous dissipation) and different channel geometries at the macroscale. An excellent review on the solution of the Graetz problem at the macroscale can be found elsewhere [4].

As the ratio of the mean-free-path ($\lambda$) to the characteristic length of the flow ($L$)–which is known as the Knudsen number ($Kn=\lambda/L$) – increases, the continuum approach fails to be valid, and the fluid modeling moves from continuum to molecular model. For the $Kn$ number varying between 0.01 and 0.1 (which corresponds to the flow of the air at standard atmospheric conditions through the channel that has the characteristic length of 1–10 μm), the regime is known as the slip-flow regime and the continuum modeling together with the slip-velocity and the temperature-jump boundary conditions (the rarefaction effect) are valid [5]. More recently, the Graetz problem has also been extended to study the microscale flows by including the rarefaction effect both analytically [6–12] and numerically [13–15].

The characteristic lengths of the microchannels are very small, therefore viscous forces dominate inertial forces leading to a low $Re$ (i.e. $Re \ll 1$) and a low $Pe$ ($Pe=RePr$). For flows with a small Peclet number, the axial conduction term cannot be neglected, since the characteristic time of the convection and the diffusion becomes comparable, and the convection term no longer dominates the conduction term in the longitudinal direction. The Graetz problem with the inclusion of the axial conduction term has been an interesting problem due to the presence of the non self-adjoint eigenvalue problem. Accordingly, the linearly independent eigenfunctions become non-orthogonal [16]. This interesting problem has been studied by many researchers for macrochannels both analytically [17–26] and computationally [27,28] for more than three decades ago. More recently, Hadjiconstantinou and Simek [29] studied the effect of axial conduction for thermally fully-developed flows in micro and nano channels; and Jeong and Jeong [30] studied the effect of axial conduction together with viscous dissipation in slit channels with micro spacing for thermally developing flow. Çetin et al. [31] studied the same problem for a microtube numerically. Dutta et al. [32] and Horiuchi et al. [33] studied the thermal characteristics of mixed electroosmotic and pressure-driven microflows with the axial conduction.

This present study extends the Graetz problem to include the rarefaction effect, viscous dissipation term, and axial conduction in

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**Nomenclature**

| | |
|---|---|
| $A_m$ | coefficients of eigenfunctions |
| $Br$ | Brinkman number ($\mu u_m^2/q_w R$) |
| $C$ | matrix defined in Eq. (29) |
| $D$ | matrix defined in Eq. (30) |
| $F_m$ | eigenfunctions |
| $FT$ | thermal accommodation factor |
| $k$ | thermal conductivity |
| $\tilde{K}$ | matrix defined in Eq. (10) |
| $Kn$ | Knudsen number ($\lambda/L$) |
| $\tilde{L}$ | matrix defined in Eq. (9) |
| $\tilde{N}$ | matrix defined in Eq. (9) |
| $Nu_{\bar{x}}$ | Local Nusselt number ($h_{\bar{x}}D/k$) |
| $Pe$ | Peclet number ($RePr$) |
| $Pr$ | Prandtl number ($\nu/\alpha$) |
| $q_w$ | wall heat flux |
| $r$ | radial coordinate |
| $\bar{r}$ | non-dimensional radial coordinate |
| $R$ | tube radius |
| $Re$ | Reynolds number ($\rho u_m D/\mu$) |
| $T$ | temperature |
| $T_i$ | inlet temperature |
| $u$ | velocity |
| $\bar{u}$ | non-dimensional velocity |
| $u_m$ | mean velocity |
| $x$ | axial coordinate |

*Greek Letters*

| | |
|---|---|
| $\beta_m$ | eigenvalues |
| $\gamma$ | specific heat ratio |
| $\eta$ | non-dimensional radial coordinate |
| $\theta$ | dimensionless temperature |
| $\theta_\infty$ | fully-developed temperature |
| $\kappa$ | parameter defined in Eq. (34) |
| $\lambda$ | mean-free-path |
| $\mu$ | viscosity |
| $\xi$ | non-dimensional axial coordinate |
| $\rho_s$ | slip radius |
| $\phi$ | dimensionless temperature |

the fluid for constant-wall-heat-flux boundary condition. By defining the appropriate non-dimensional parameters, the given problem is formulated in a similar form with its macroscale counterpart. The temperature distribution is determined analytically by using the general eigenfunction expansion and is obtained in terms of confluent hypergeometric functions. The effects of the rarefaction, axial conduction and viscous dissipation on the local $Nu$ are discussed in terms of dimensionless parameters such as the $Kn$, $Pe$ and $Br$.

## 2. Analysis

The steady-state, hydrodynamically-developed flow with a constant temperature, $T_i$, flows into the microtube with the constant heat flux at the wall, as shown in Fig. 1. By introducing the following dimensionless parameters,

$$\bar{r} = \frac{r}{R}, \quad \bar{x} = \frac{x}{PeR}, \quad \theta = \frac{T - T_i}{q_w R/k}, \quad \bar{u} = \frac{u}{u_m}, \quad Pe = RePr,$$
$$Br = \frac{\mu u_m^2}{q_w R}, \tag{1}$$

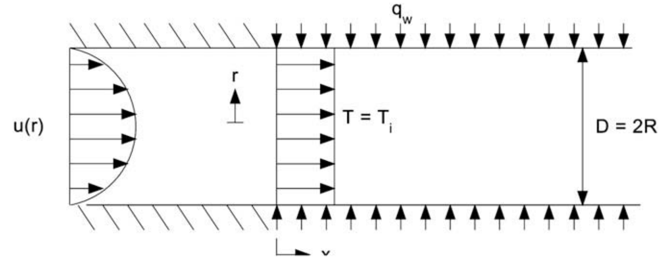


**Fig. 1.** Geometry of the problem.

the governing energy equation, including the axial conduction and the viscous dissipation term, and the corresponding boundary conditions can be written as [13]

$$\frac{\bar{u}}{2}\frac{\partial\theta}{\partial\bar{x}} = \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\theta}{\partial\bar{r}}\right) + \frac{1}{Pe^2}\frac{\partial^2\theta}{\partial\bar{r}^2} + \frac{32Br}{(1+8Kn)^2}\bar{r}^2, \tag{2}$$

$$\theta = 0 \quad \text{at} \quad \bar{x} = 0, \tag{3}$$

$$\frac{\partial\theta}{\partial\bar{r}} = 0 \quad \text{at} \quad \bar{r} = 0, \tag{4}$$

$$\frac{\partial\theta}{\partial\bar{r}} = 1 \quad \text{at} \quad \bar{r} = 1, \tag{5}$$

$$\theta \to \theta_\infty \quad \text{at} \quad \bar{x} \to \infty, \tag{6}$$

where $\bar{u}$ is the dimensionless fully-developed velocity profile for the slip-flow regime defined as [31],

$$\bar{u} = \frac{2(1 - \bar{r}^2 + 4Kn)}{1 + 8Kn}, \tag{7}$$

and $\theta_\infty$ is the dimensionless fully-developed temperature profile which can be determined by applying the similar procedure to that for a macrochannel flow [34], and the solution can be written in matrix form as,

$$\tilde{M} = \begin{bmatrix} 1/4 & -1 & 7/24 & -4 & -8 \\ 2 & -4 & 1 & 16 & 32 \\ 3 & -14 & 14 & -64 & -128 \\ 18 & -36 & 11 & 192 & -384 \\ 18 & -108 & 41 & -576 & -1152 \\ 3 & -6 & 2 & 48 & -96 \\ 0 & 2 & -1 & 16 & 32 \\ 3 & -30 & 13 & -192 & -384 \end{bmatrix} \tag{8}$$

$$\tilde{N} = \begin{bmatrix} \bar{r}^4 \\ \bar{r}^2 \\ 1 \\ \bar{x} \\ 1/Pe^2 \end{bmatrix}, \quad \tilde{L} = -\frac{1}{(1+8Kn)^4}\begin{bmatrix} 1 \\ 2Br \\ 2Kn \\ 8BrKn \\ 8/3Kn^2 \\ 128/3BrKn^2 \\ -1024Kn^4 \\ 128/3Kn^3 \end{bmatrix} \tag{9}$$

$$\tilde{K} = \tilde{L}\left(\tilde{M}\tilde{N}\right), \tag{10}$$

$$\theta_\infty = \sum_{i=1}^{9}\tilde{K}_i. \tag{11}$$